Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
Local Class Field Theory: Addenda

Reference. Neukirch, IV.4-IV.6; also V.1 for the first section.

The local existence theorem. We've seen two ways to derive the local reciprocity law, but I somehow managed not to prove the local existence theorem. Fortunately, it's easy. (Again, this argument only works in characteristic 0.)

Theorem 1. *Let K be a finite extension of $\mathbb{Q}_p$. Then every open subgroup of K^* of finite index is of the form $\mathrm{Norm}_{L/K} L^*$ for some finite (abelian) extension L of K.*

Recall that $\mathrm{Norm}_{L/K} L^* = \mathrm{Norm}_{M/K} M^*$ for $M = L \cap K^{\mathrm{ab}}$ by the local reciprocity law, so we don't have to make sure L is abelian.

Proof. Let N be an open subgroup of K^* of index n. Then $(K^*)^n \subseteq N$, and it suffices to prove that $(K^*)^n$ contains $\mathrm{Norm}_{L/K} L^*$ for some L. Let ζ_n be a primitive n-th root of unity, let μ_n be the group generated by ζ_n, and put $K_1 = K(\zeta_n)$. By Kummer theory, there is a canonical isomorphism between $\mathrm{Hom}(\mathrm{Gal}(K_1^{\mathrm{ab}}/K_1), \mu_n)$ and $K_1^*/(K_1^*)^n$. But the latter is finite for any local field K_1 (exercise), so if we let L be the compositum of all $\mathbb{Z}/n\mathbb{Z}$-extensions of K_1, then L/K_1 is a finite extension. Now $\mathrm{Gal}(L/K_1) \cong K_1^*/\mathrm{Norm}_{L/K_1} L^*$ is a group of exponent n, so $(K_1^*)^n \subseteq \mathrm{Norm}_{L/K_1} L^*$. On the other hand,

$$\# K_1^*/(K_1^*)^n = \# \mathrm{Gal}(L/K_1) = \# K_1^*/\mathrm{Norm}_{L/K_1} L^*,$$

so $(K_1^*)^n = \mathrm{Norm}_{L/K_1} L^*$. In particular, $(K^*)^n$ contains $\mathrm{Norm}_{K_1/K}(K_1^*)^n = \mathrm{Norm}_{L/K} L^*$, as desired. □

More on abstract class field theory.

First of all, the claim that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is isomorphic to $\widehat{\mathbb{Z}}$ is false; that group is of course $\widehat{\mathbb{Z}}^*$. But it has a natural quotient isomorphic to $\widehat{\mathbb{Z}}$; it's easiest to construct this factor by factor, by exhibiting a surjection $\mathbb{Z}_p^* \to \mathbb{Z}_p$. (Otherwise, put, $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$ has a subextension with Galois group $\mathbb{Z}_p$.) For $p \neq 2$, $\mathbb{Z}_p^*$ can be written as the product of the group of $(p-1)$-st roots of unity with the subgroup $1 + p\mathbb{Z}_p$, and the latter is isomorphic to $p\mathbb{Z}_p$ via the logarithm map. For $p = 2$, $\mathbb{Z}_p^*$ can be written as the product of $\{\pm 1\}$ with the subgroup $1 + 4\mathbb{Z}_2$, which again is isomorphic to $4\mathbb{Z}_2$ via the logarithm map.

Second, the proof of the multiplicativity of the reciprocity map was seriously broken. Having finally deciphered the proof in Neukirch, I'll give a version of that instead.

Recall notation: L/K is a finite extension of finite extensions of k, H is the semigroup of $g \in \mathrm{Gal}(L^{\mathrm{unr}}/K)$ such that $d_K(g) \in \mathbb{N}$ (that's positive integers, not nonnegative), and $r' : H \to A_K/\mathrm{Norm}_{L/K} A_L$ is defined as follows. For $g \in \mathrm{Gal}(L^{\mathrm{unr}}/K)$, let M be the fixed field of g, let π_M be a uniformizer of M, and set $r'(g) = \mathrm{Norm}_{M/K}(\pi_M)$.

Given $g_1, g_2 \in H$, set $g_3 = g_1 g_2$; we want to show that $r'(g_1)r'(g_2) = r'(g_3)$. Let M_i be the fixed field of g_i, let π_i be a uniformizer of π_i, and put $\rho_i = r(g_i) = \mathrm{Norm}_{M_i/K}(\pi_i)$; since $v_K(\rho_i) = d_K(g_i)$, the ratio $\rho_1\rho_2/\rho_3$ is a unit; we want to show that it is in $\mathrm{Norm}_{L/K}(U_L)$.

Again, the trouble with $\rho_1\rho_2/\rho_3$ is that it is made up of norms from different fields, and to make progress we need to rewrite it more uniformly. Choose $\phi \in \mathrm{Gal}(L^{\mathrm{unr}}/K)$ with $d_K(\phi) = 1$, and put $d_i = d_K(g_i)$; then we can write $g_i = \phi^{d_i} h_i^{-1}$ for some h_i with $d_K(h_i) = 0$, that is, $h_i \in \mathrm{Gal}(L^{\mathrm{unr}}/K^{\mathrm{unr}})$.

Proposition 2. *Let M be the fixed field of some $h \in \mathrm{Gal}(L^{\mathrm{unr}}/K)$ with $d_K(h) = n$ a nonnegative integer, and suppose $\phi \in H$ satisfies $d_K(\phi) = 1$. Then for any $x \in A_M$,*

$$\mathrm{Norm}_{M/K}(x) = \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(x x^{\phi} \cdots x^{\phi^{n-1}}).$$

Put

$$\sigma_i = \pi_i \pi_i^{\phi} \cdots \phi_i^{\phi^{d_i - 1}};$$

then if $u = \sigma_1\sigma_2/\sigma_3$, we have $u \in U_{L^{\mathrm{unr}}}$ and $\rho_1\rho_2/\rho_3 = \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(u)$. Now

$$\frac{u^{\phi}}{u} = \frac{\sigma_1^{\phi}\sigma_2^{\phi}\sigma_3}{\sigma_1\sigma_2\sigma_3^{\phi}} = \frac{\pi_1^{\phi^{d_1}}\pi_2^{\phi^{d_2}}\pi_3}{\pi_1\pi_2\pi_3^{\phi^{d_3}}} = \frac{\pi_1^{h_1}\pi_2^{h_2}\pi_3}{\pi_1\pi_2\pi_3^{h_3}}$$

since π_i is fixed by $g_i = \phi^{d_i} h_i^{-1}$. The relationship among the h_i is $\phi^{d_1} h_1^{-1} \phi^{d_2} h_2^{-1} = \phi^{d_3} h_3^{-1}$, that is, $h_3 = h_2 \phi^{-d_2} h_1 \phi^{d_2}$.

Pick a single uniformizer π_L of L; then we can write $\pi_i = v_i \pi_L$ for some unit $v_i \in L^{\mathrm{unr}}$, and

$$\frac{u^{\phi}}{u} = \frac{v_1^{h_1} v_2^{h_2} v_3}{v_1 v_2 v_3^{h_3}} \frac{\pi_L^{h_1}\pi_L^{h_2}}{\pi_L \pi_L^{h_3}}.$$

The first term is clearly a product of expressions of the form $y_i^{g_i}/y_i$ for $y_i \in U_{L^{\mathrm{unr}}}$ and $g_i \in \mathrm{Gal}(L^{\mathrm{unr}}/K^{\mathrm{unr}})$. On the other hand, since π_L and $\pi_L^{h_1}$ are invariant under ϕ, we get

$$\begin{aligned}\frac{\pi_L^{h_1}\pi_L^{h_2}}{\pi_L\pi_L^{h_3}} &= \frac{\pi_L^{h_1}\pi_L^{h_2}}{\pi_L^{\phi^{d_2}}\pi_L^{h_2\phi^{-d_2}h_1\phi^{d_2}}} \\ &= \frac{\pi_L^{h_2}}{\pi_L^{d_2}}\left(\frac{\pi_L^{d_2}}{\pi_L^{h_2}}\right)^{\phi^{-d_2}h_1\phi^{d_2}}.\end{aligned}$$

Thus $\frac{u^{\phi}}{u}$ is the product of expressions of the form $y_i^{g_i}/y_i$ for $y_i \in U_{L^{\mathrm{unr}}}$ and $g_i \in \mathrm{Gal}(L^{\mathrm{unr}}/K^{\mathrm{unr}})$. What we need to show is that $\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(U_L) \in \mathrm{Norm}_{L/K}(U_L)$; that will follow from the following lemma.

Lemma 3. *If $x \in U_{L^{\mathrm{unr}}}$ has the property that $x^{\phi}/x = \prod_i y_i^{g_i}/y_i$ for some $y_i \in U_{L^{\mathrm{unr}}}$ and $g_i \in \mathrm{Gal}(L^{\mathrm{unr}}/K^{\mathrm{unr}})$, then $\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(x)$ belongs to $\mathrm{Norm}_{M/K} U_M$ for every finite subextension M of L^{unr}/K.*

Proof. (This is Neukirch, Lemma IV.5.4.) It suffices to consider M containing x and each y_i, and also containing L. Let $n = [M : K]$, and let F be the fixed field of $\sigma = \phi^n$. Let G be the degree n unramified extension of G, i.e., the fixed field of σ^n. By the vanishing of

$H_T^0(\mathrm{Gal}(G/F), U_G)$, we can write $x = \mathrm{Norm}_{G/F}(\tilde{x})$ and $y_i = \mathrm{Norm}_{G/F}(\tilde{y}_i)$ for some $\tilde{x}, \tilde{y}_i \in U_G$. Put

$$\alpha = \frac{\tilde{x}^{\phi}}{\tilde{x}} \prod_i \frac{\tilde{y}_i{}^{g_i}}{\tilde{y}_i};$$

then $\mathrm{Norm}_{G/F}(\alpha) = 1$. By the vanishing of $H_T^{-1}(\mathrm{Gal}(G/F), U_G)$, that means $\alpha = w^{\sigma}/w$ for some $w \in G$. That means we can write

$$\frac{\tilde{x}^{\phi}}{\tilde{x}} = \frac{(ww^{\phi} \cdots w^{\phi^{n-1}})^{\phi}}{ww^{\phi} \cdots w^{\phi^{n-1}}} \prod_i \frac{\tilde{y}_i{}^{g_i}}{\tilde{y}_i}.$$

Apply $\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}$ to both sides. This kills off $\tilde{y}_i{}^{g_i}/\tilde{y}_i$ for each i, leaving

$$\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(\tilde{x})^{\phi-1} = \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(ww^{\phi} \cdots w^{\phi^{n-1}})^{\phi-1}.$$

Put $z = \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(\tilde{x}/(ww^{\phi} \cdots w^{\phi^{n-1}}))$; then the previous equation reads $z^{\phi} = z$, which means $z \in U_K$. We then have

$$\begin{aligned}
\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(x) &= \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(\mathrm{Norm}_{G/F}(\tilde{x})) \\
&= \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(\tilde{x}\tilde{x}^{\phi^n} \cdots \tilde{x}^{\phi^{n(n-1)}}) \\
&= \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(zz^{\phi^n} \cdots z^{\phi^{n(n-1)}}) \, \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(ww^{\phi} \cdots w^{\phi^{n^2-1}}) \\
&= z^n \, \mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(tt^{\phi} \cdots t^{\phi^{n-1}}),
\end{aligned}$$

if we put $t = ww^{\phi^n} \cdots w^{\phi^{n(n-1)}}$. Now applying the above proposition, and noting that trivially $z^n = \mathrm{Norm}_{F/K}(z)$, we get $\mathrm{Norm}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(x) = \mathrm{Norm}_{M/K}(z) \, \mathrm{Norm}_{F/K}(t) \in \mathrm{Norm}_{M/K} U_M$, as desired. $\square$