The Dedekind Zeta Function and the Class Number Formula
Math 254B Final Paper

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1 Introduction

The aim of this paper is to prove that the Dedekind zeta function for a number field has a meromorphic continuation to the complex plane, obtaining the analytic class number formula with it, and to present some of the applications of these results. In particular, It will be shown its use in proving Dirichlet's prime number theorem and in calculating the class number of quadratic fields.

The proof of the class number formula and analytic continuation of the zeta function will be complete except for some parts involving technical calculations which would not add much to the number theory concepts with which this paper tries to deal. It will mainly follow the proof in the book by Neukirch [2].

We will begin by demonstrating the use of the class number formula to find an expression for the class number of a quadratic field and prove Dirichlet's prime number theorem. The proof of the main result will come after this.

2 Two applications

2.1 Some definitions and main results

The central object studied here is the Dedekind zeta function:

Definition 2.1. The Dedekind zeta function of a number field K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

with the sum running over all the integral ideals of K.

In this paper $\mathfrak{N}(\mathfrak{a})$ will represent the absolute norm of an ideal, and K will always be a general number field except when otherwise stated. Although the series used in the definition converges only for $Re(s) > 1$, the result we will prove later states that $\zeta_K(s)$ admits a holomorphic continuation to the whole complex plane except the point $s = 1$, where the function has a simple pole. We have the

Theorem 2.2. ***(Analytic Class Number Formula)***

$$\mathrm{Res}_{s=1}(\zeta_K(s)) = \frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_K|}}hR$$

Here, R represents the regulator of K, defined in [2], chapter 1, 7; d_K denotes the discriminant of K.

In the case $K = \mathbb{Q}$, the zeta function is:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

called the Riemann zeta function. It is generalized by the *Dirichlet L-series*:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where χ is any character $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$. The analytic continuation for this series will not be proved in this paper, but it can be found in [2] (the proof resembles closely the one that will be carried out here). In fact, $L(\chi, s)$ can be extended to a holomorphic function in the whole complex plane, including $s = 1$, when χ is a nontrivial character (for χ trivial, $L(\chi, s)$ is the Riemann zeta function). We will use this in a moment. We will also need the famous Euler's identity:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1}$$

which in the case of $K = \mathbb{Q}$ takes the form

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

and that can be generalized for the Dirichlet L-series:

$$L(\chi, s) = \prod_{p} \left(1 - \frac{\chi(s)}{p^s}\right)^{-1} \tag{2.1}$$

where χ is any character.

Euler's product is well known for being an analytical form of the theorem of unique decomposition of ideals into prime ideals in a number field. It can also help us relate the zeta function for a field to the Riemann zeta function, as we do now for a quadratic number field.

2.2 The class number of quadratic number fields

In this section, $K = \mathbb{Q}(\sqrt{D})$ will represent a quadratic number field, with D squarefree. It is not difficult to show that a prime p in $\mathbb{Z}$ splits in K depending on the value of the Legendre symbol $(\frac{d_K}{p})$ in the following way:

$$p \text{ is...} \begin{cases} \text{...unsplit} & \text{if } (\frac{d_K}{p}) = -1 \\ \text{...split into two different primes} & \text{if } (\frac{d_K}{p}) = 1 \\ \text{...split in the form } p = \mathfrak{p}^2 & \text{if } (\frac{d_K}{p}) = 0 \end{cases}$$

Result (8.5) in chapter I of [2] gives the proof of this except when $p \mid 2d_K$. In this case, if $p \mid d_K$ then $p = (p, \sqrt{D})^2$ and if $p = 2$, then:

$$(2) = \begin{cases} (2) & \text{if } D \equiv 5 (\mathrm{mod}\, 8) \quad (\text{so } (\frac{d_K}{2}) = -1) \\ (2, \frac{1+\sqrt{D}}{2})(2, \frac{1-\sqrt{D}}{2}) & \text{if } D \equiv 1 (\mathrm{mod}\, 8) \quad (\text{so } (\frac{d_K}{2}) = 1) \\ (2, 1+\sqrt{D})^2 & \text{if } D \equiv 3, 7 (\mathrm{mod}\, 8) \quad (\text{so } (\frac{d_K}{2}) = 0) \\ (2, \sqrt{D})^2 & \text{if } D \equiv 2, 6 (\mathrm{mod}\, 8) \quad (\text{so } (\frac{d_K}{2}) = 0) \end{cases}$$

Hence, we know all the primes in K, together with their norm, by relating them to primes in $\mathbb{Z}$. This lets us write the zeta function for $K = \mathbb{Q}(\sqrt{D})$ and for any $s > 1$ as (in the following, consider $\mathfrak{p}$ to run over all primes of K and p to run over all primes of $\mathbb{Z}$)

$$\begin{aligned} \zeta_K(s) =& \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s})\right)^{-1} = \\ =& \prod_{(\frac{d_K}{p})=1} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \prod_{(\frac{d_K}{p})=-1} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \mid d_K} \left(1 - \frac{1}{p^s}\right)^{-1} = \\ =& \prod_{(\frac{d_K}{p})=1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 + \frac{1}{p^s}\right)^{-1} \prod_{(\frac{d_K}{p})=-1} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \mid d_K} \left(1 - \frac{1}{p^s}\right)^{-1} \end{aligned}$$

So we have:

$$\zeta_K(s) = \zeta(s) L(d_K, s) \quad \text{for } s > 1$$

where

$$\begin{aligned} L(d_K, s) :=& \prod_{(\frac{d_K}{p})=1} \left(1 + \frac{1}{p^s}\right)^{-1} \prod_{(\frac{d_K}{p})=-1} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \mid d_K} 1 = \\ =& \prod_p \left(1 - \frac{(\frac{d_K}{p})}{p^s}\right)^{-1} = \sum_{p=1}^{\infty} \frac{(\frac{d_K}{n})}{n^s} = \\ =& L((\frac{d_K}{n}), s) \end{aligned} \tag{2.2}$$

This last equality is obtained from Euler's product 2.1; $L(d_K, s)$ is a particular instance of the Dirichlet L-series for the character $\chi(n) = (\frac{d_K}{n})$.

The class number formula gives us the residue of $\zeta_K(s)$ in $s = 1$. As $\zeta(s)$ has residue 1 in this point, the residue of $\zeta_K(s)$ must equal the value of $L(d_K, s)$ at $s = 1$ ($L(d_K, s)$ can be extended to a holomorphic function in $\mathbb{C}$, as was pointed out before, and it has a value at $s = 1$, defined by continuity. Equation 2.1 holds for all $s \neq 1$).

So $L(d_K, 1)$ *is precisely the value given by the class number formula.* If it can be computed then we can calculate the value of the class number for K (remember we are considering only quadratic fields). Can it be calculated using the same series that defines it, putting $s = 1$?. The answer is yes, but it is not something completely evident: by definition, the value $L(d_K, 1)$ is given by the limit of the series at $s = 1$, so we have to prove that this limit coincides with the value of the series in $s = 1$, this is, that the function defined by the series is continuous in $s = 1$. This is in fact true, as is proved in [1] by rewriting the series for $L(d_K, s)$ in a way that makes evident the continuity at $s = 1$. So now let's calculate $L(d_K, 1)$ from the series that appears in (2.2).

As a first example, take $K = \mathbb{Q}(i)$, with $d_K = -4$. We get:

$$L(d_K, 1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

which can be calculated by noting that

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2} = \int_0^1 (1 - x^2 + x^4 - x^6 + \cdots)dx = L(d_K, 1)$$

(A power series can be integrated term by term within its radius of convergence). As $\mathbb{Q}(i)$ has one pair of complex conjugate embeddings, has four roots of unity and regulator $R = 1$, the class number formula gives $L(d_K, 1) = (2\pi/4\sqrt{4})h = (\pi/4)h$. Hence, $h = 1$ in this case.

For $K = \mathbb{Q}(\sqrt{5})$ we can do something analogous:

$$\begin{aligned} L(d_K, 1) = (1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}) + (\frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9}) = \\ = \int_0^1 (1 - x - x^2 + x^3)(1 + x^5 + x^10 + x^15 + \cdots) \\ = \int_0^1 \frac{1 - x - x^2 + x^3}{1 - x^5} dx \quad (2.3) \end{aligned}$$

This is an integral that can be calculated using well-known methods, or even only approximated, in order to compare it to the value given by the class number formula. Knowing that we have two complex embeddings, two roots of unity and that $R = \log\left(\frac{\sqrt{5}+1}{2}\right)$ (the logarithm of the fundamental unit), one gets:

$$L(d_K, 1) = \frac{2^2}{2\sqrt{5}} h \log\left(\frac{\sqrt{5}+1}{2}\right) = \frac{2}{\sqrt{5}} \log\left(\frac{\sqrt{5}+1}{2}\right) h$$

As h is an integer, a close enough approximation to the integrals lets us conclude that $h = 1$ (note that, curiously, this gives us the exact value of the integral).

In general, following the same procedure we obtain:

$$L(d_K, 1) = \int_0^1 \frac{f_{d_K}(x)dx}{1 - x^{|d_K|}} \quad \text{, where } f_{d_K}(x) = \sum_{t=1}^{|d_K|} x^{t-1} \left(\frac{d_K}{t}\right)$$

This integral can be approximated as before to obtain h. Thus, a formula for the class number of a quadratic field K is given by:

$$h = \frac{1}{R} \frac{w\sqrt{|d_K|}}{2^{r_1}(2\pi)^{r_2}} \int_0^1 \frac{f_{d_K}(x)dx}{1 - x^{|d_K|}}$$

As a curiosity, I will include a more explicit form of this formula for the class number taken from [3]. The class number h of a quadratic field can be written as:

$$h = \begin{cases} -\frac{1}{2\log\eta} \sum_{r=1}^{d_K-1} (\frac{d_K}{r}) \sin(\frac{\pi r}{d_K}) & \text{for } d_K > 0 \\ -\frac{w}{2|d_K|} \sum_{r=1}^{|d_K|-1} (\frac{d_K}{r}) r & \text{for } d_K < 0 \end{cases} \tag{2.4}$$

where η is the fundamental unit of the field of positive discriminant, (d_K/r) denotes the Kronecker symbol and w is the number of roots of unity in the field.

2.3 Dirichlet's Prime Number Theorem

This well known result states that

Theorem 2.3. *If a and m are coprime natural numbers, then there are infinitely many primes in the sequence $(a + kn)$, with $k \in \mathbb{N}$*

Its proof can surprisingly be reduced to the statement that $L(\chi, 1) \neq 0$ for any χ non-trivial character modulo m by writing:

$$\begin{aligned} \log L(\chi, s) &= -\sum \log(1 - \frac{\chi(p)}{p^s}) = \left(\text{use } \log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } |x| < 1\right) \\ &= \sum_p \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}} = \sum_p \frac{\chi(p)}{p^s} + \sum_{m=2}^{\infty} \sum_p \frac{\chi(p^m)}{mp^{ms}} = \\ &= \sum_p \frac{\chi(p)}{p^s} + g(s) \end{aligned}$$

where $g(s)$ is defined by a series which is convergent for $\mathrm{Re}(s) > 1/2$ (note that it starts in $m = 2$), so it is bounded near $s = 1$.

Now sum over all characters χ modulo m and multiply by $\chi(a^{-1})$, with a any natural number prime to m:

$$\sum_{\chi} \chi(a^{-1}) \log L(\chi, s) = \sum_{\chi} \sum_{p} \frac{\chi(a^{-1})}{p^s} + g(s) =$$

(interchange the summation and break $\sum_{p}$ into classes $\bmod\, m$)

$$= \sum_{b=1}^{m} \sum_{\chi} \chi(a^{-1}b) \sum_{p \equiv b\,(m)} \frac{1}{p^s} + g(s) =$$

$$= \sum_{p \equiv a\,(m)} \frac{\phi(m)}{p^s} + g(s)$$

as the sum in χ has the value

$$\sum_{\chi} \chi(a^{-1}b) = \begin{cases} 0 & , a \neq b \\ \phi(m) & , a = b \end{cases}$$

Now that we have the result

$$\sum_{\chi} \chi(a^{-1}) \log L(\chi, s) = \sum_{p \equiv a\,(m)} \frac{\phi(m)}{p^s} + g(s) \tag{2.5}$$

knowing that $L(\chi, s) \neq 0$ for χ nontrivial would tell us that the left hand side tends to $+\infty$ as s tends to 1, as everything there is finite except for χ_0 =trivial character $\bmod\, m$, for which:

$$\log L(\chi_0, s) = \sum_{p|m} \log(1 - p^{-s}) + \log \zeta(s) \quad ,$$

expression which tends to $+\infty$ as $s \to 1$. The right hand side then tends to $+\infty$ too, so the sum cannot consist of finitely many terms and we obtain Dirichlet's prime number theorem.

So the problem is to show that $L(\chi, 1) \neq 0$ for a nontrivial character χ. This can be done in many ways, of which the most common is to relate Dirichlet L-series to the zeta function of some field. We have already done this in a particular case when we said that $L(\chi, 1)$ =(expression from the class number formula) when $\chi(n) = (d_k/n)$. Of course, this says that $L(\chi, 1) \neq 0$ for this χ. Can we find something like this for any character $\bmod\, m$?. It can be done, and in different ways. In [2], Neukirch proves that:

$$\zeta_K(s) = G(s) \prod_{\chi} L(\chi, s) \quad \text{for } K = \mathbb{Q}(\zeta_m)$$

where $G(s)$ is some bounded function defined in $\mathbb{C}$. We see then from the analytic continuation of ζ_K that $L(\chi, 1) \neq 0$ for χ nontrivial (the only pole of the right hand side is provided by $L(\chi_0, s)$ at $s = 1$, which is simple as the one for $\zeta_K(s)$).

However, that $L(\chi, 1) \neq 0$ can also be deduced using our previous result that $L(d_K, s) \neq 0$ for K a quadratic number field. This is carried out in [1], chapter X, section 11, where it is proved that $L(\chi, s) = L((\frac{d_K}{n}), s)f(s)$ for f a nowhere zero function and χ a real nontrivial character. For a complex nontrivial character in can be seen that if $L(\chi, 1) = 0$ then the left hand side of 2.5 tends to $-\infty$ as $s \to 1$, which is impossible since the right hand side is positive.

3 Analytic Continuation of the Dedekind Zeta Function and the Class Number Formula

3.1 Introduction

The object studied in this section is the Dedekind zeta function we defined in 2.1. It is a natural generalization of the Riemann zeta function, which was originally studied as a tool that gives useful information on number theory problems such as the distribution of prime numbers, and extends the methods used in $\mathbb{Q}$ to a general number field. Hecke L-series (not defined here) are, analogously a generalization of Dirichlet L-series.

All of these are initially defined as complex functions on certain region of $\mathbb{C}$, and each of them can be later proved to have a meromorphic extension to all of $\mathbb{C}$ and to satisfy a functional equation relating their values in s to those in $1-s$. The overall structure of the proof of these results is the same for all of them, and uses the Mellin transform as a fundamental tool:

Definition 3.1. For a continuous function $f : \mathbb{R}_+^* \longrightarrow \mathbb{C}$, the Mellin transform of f is:

$$L(f, s) = \int_0^\infty (f(y) - f(\infty))\, y^s \frac{dy}{y}$$

provided that both the integral and the limit $f(\infty) = \lim_{y\to\infty} f(y)$ exist.

The next theorem is a central part of the proof. The statement is taken from the book of Neukirch [2]:

Theorem 3.2. *Let $f, g : \mathbb{R}_+^* \longrightarrow \mathbb{C}$ be continuous functions such that*

$$f(y) = a_0 + O(e^{-cy^\alpha}), \quad g(y) = b_0 + O(e^{-cy^\alpha})$$

for $y \to \infty$, with positive constants c, α. If these functions satisfy the equation

$$f\left(\frac{1}{y}\right) = Cy^k g(y)$$

for some real number $k > 0$ and some complex number $C \neq 0$, then one has:

1. *The integrals $L(f,s)$ and $L(g,s)$ converge absolutely and uniformly if s varies in an arbitrary compact domain contained in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > k\}$. They are therefore holomorphic functions on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > k\}$. They admit holomorphic continuations to $\mathbb{C} \setminus \{0, k\}$.*

2. *They have simple poles at $s = 0$ and $s = k$ with residues*

$$\begin{aligned} \mathrm{Res}_{s=0} L(f,s) &= -a_0, \quad \mathrm{Res}_{s=0} L(f,s) = Cb_0 \\ \mathrm{Res}_{s=0} L(g,s) &= -b_0, \quad \mathrm{Res}_{s=0} L(g,s) = C^{-1}a_0 \end{aligned}$$

3. *They satisfy the functional equation*

$$L(f,s) = CL(g, k-s)$$

3.2 Structure of the Proof

The proof to come will follow the steps below, which are shared with proofs of the analytic continuation of other types of zeta functions:

1. First, the function is defined by a series that converges only for $s/\,\mathrm{Re}\, s > 1$.

2. Then, the expression for the series is manipulated introducing the gamma function or a suitable generalization, to express it as the Mellin transform of some theta series (defined later) or related function. The modified zeta function is called *completed zeta function*.

3. It is proved then that the function involving the theta series satisfies the conditions of theorem 3.2.

4. This theorem is the one that finally justifies the extension and functional equation for the completed zeta function.

5. The functional equation for the original zeta function is deduced from this, using known functional equations for the Γ function involved.

This process also leads to the determination of the residues of the zeta function in its poles; the formula for the residue of the Dedekind zeta function 2.1 is known as the *analytic class number formula*.

3.3 The Gamma Function

As explained above, we will need the real gamma function and its generalization to number fields in what follows.

Definition 3.3. The real gamma function is defined for $\mathrm{Re}(s) > 0$ by the integral:

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$$

The following properties will be relevant for our purpose:

Proposition 3.4. *1. The gamma function is analytic and admits a meromorphic continuation to all of $\mathbb{C}$.*

2. It is nowhere zero and its only poles happen at $s = 0, -1, -2\ldots$ They are all simple; the residue at $-n$ is $(-1)^n/n!$.

3. It satisfies the functional equations:

(a) $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$

(b) $\Gamma(s)\Gamma(s+\frac{1}{2}) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$

(c) $\Gamma(s+1) = s\Gamma(s)$

The functional equation in 3.4, (3a) can be proved without first extending the gamma function (for the values of s for which it would make sense); this would also give a way of proving that it has a meromorphic continuation to the complex plane, and would yield its poles. The other functional equations will be used when we prove the one satisfied by the higher dimensional gamma function, a generalization of the present one.

To define this generalized gamma function we will eventually write it as the same integral we used for Γ, this time over some subset of the trace-zero Minkowski space of a number field that will be the analogue of $\mathbb{R}_+^*$ in this setting. To achieve this first we need to define this subset and establish some notation conventions, and then we need to fix a measure on it.

Let K be a number field. Denote $G = \mathrm{Gal}(K|\mathbb{Q})$. We will denote by $\mathbf{C}$ the associated Minkowski space and by $\mathbf{R}$ the trace-zero Minkowski space of K. We will denote, for $x = (x_\tau)_{\tau\in G} \in \mathbf{C}$, $\overline{x} = (\overline{x}_{\overline{\tau}})_{\tau\in G}$, as usual. We will also write:

$$\mathbf{R}_\pm = \{x \in \mathbf{R} \mid x = \overline{x}\} \qquad \text{(the analogue of } \mathbb{R}\setminus\{0\}) \tag{3.1}$$

$$\mathbf{R}_+^* = \{x \in \mathbf{R}_\pm \mid x > 0\} \qquad \text{(analogue of } \mathbb{R}_+^*) \tag{3.2}$$

$$\mathbf{H} = \mathbf{R}_\pm + i\mathbf{R}_+^* \qquad \text{(analogue of the upper half plane of } \mathbb{C}) \tag{3.3}$$

The $x > 0$ in the second line is taken to mean that every component of x is greater than zero. We can also define the functions $|\quad| : \mathbf{R}^* \longrightarrow \mathbf{R}_+^*$ and $\log : \mathbf{R}_+^* \longrightarrow \mathbf{R}_\pm$, which act on each component as their real analogues. Recall that in the Minkowski space we also have a scalar product given by $\langle x, y\rangle = \sum_\tau x_\tau \overline{y}_\tau$. If $z = (z_\tau)$ and $p = (p_\tau)$ are in $\mathbf{C}$, we will write

$$z^p = (z_\tau^{p_\tau})$$

provided that no component of z is real negative or 0. Here, complex exponentiation is defined in the usual way, using the principal branch of the logarithm.

Now, to fix a Haar measure in $\mathbf{R}_+^*$, denote by $\mathfrak{p}$ a general conjugation class in X (this is, $\mathfrak{p} = \{\sigma, \overline{\sigma}\}$ for some $\sigma \in G$, with either one or two elements depending on whether σ is real or complex). Define an isomorphism:

$$\Phi : \mathbf{R}_+^* \xrightarrow{\cong} \prod_{\mathfrak{p}} \mathbb{R}_+^*$$

where the component of $\Phi(x)$ corresponding to $\mathfrak{p} = \{\sigma, \overline{\sigma}\}$, for $x = (x_\tau)_{\tau \in G}$, is given by x_σ if $\mathfrak{p}$ is real and by x_σ^2 if $\mathfrak{p}$ is complex (note that for $x \in \mathbf{R}_+^*$, $x_\sigma = x_{\overline{\sigma}} \in \mathbb{R}$).

Denote by $\frac{dy}{y}$ the Haar measure on $\mathbf{R}_+^*$ induced by this isomorphism, taking the product measure $\prod_{\mathfrak{p}} \frac{dt}{t}$ in the image space. It is called the *canonical measure on* $\mathbf{R}_+^*$.

Now we are prepared for the

Definition 3.5. Given a number field K and using the same notation as before, we define its associated gamma function by the following integral, valid for all $\mathbf{s} \in \mathbf{C}$ such that all of its components have positive real part:

$$\Gamma_K(\mathbf{s}) = \int_{\mathbf{R}_+^*} \mathrm{N}(e^{-y} y^s) \frac{dy}{y}$$

where $N(\cdots)$ denotes the usual norm in the Minkowski space (the product of the components).

The problem of whether it is well defined or not is easily reduced to the study of the real gamma function using the isomorphism we defined before, which reduces the integral over $\mathbf{R}_+^*$ to a product of real integrals. The result, for $\mathbf{s} = (s_\sigma)_{\sigma \in G}$ is:

$$\Gamma_K(\mathbf{s}) = \prod_{\mathfrak{p}} \Gamma_{\mathfrak{p}}(\mathbf{s}_{\mathfrak{p}})$$

where $\mathbf{s}_{\mathfrak{p}} = s_\sigma$ if $\mathfrak{p} = \{\sigma\}$ a real embedding, $\mathbf{s}_{\mathfrak{p}} = (s_\sigma, s_{\overline{\sigma}})$ if $\mathfrak{p} = \{\sigma, \overline{\sigma}\}$ is a complex one, and:

$$\Gamma_{\mathfrak{p}}(\mathbf{s}_{\mathfrak{p}}) = \begin{cases} \Gamma(\mathbf{s}_{\mathfrak{p}}) & \text{if } \mathfrak{p} \text{ is real} \\ 2^{1-\mathrm{Tr}(\mathbf{s}_{\mathfrak{p}})} \Gamma(\mathrm{Tr}(\mathbf{s}_{\mathfrak{p}})) & \text{if } \mathfrak{p} \text{ is complex} \end{cases} \tag{3.4}$$

Note that this formula proves that Γ_K can be extended to all of $\mathbf{C}$ and gives a straightforward way to calculate its poles once we know those of the real gamma function.

Apart from this general gamma function itself, later we will need some closely related functions, namely:

$$\begin{aligned} \Gamma_K(s) = \Gamma_K(s \cdot \mathbf{1}) = 2^{(1-2s)r_2} \Gamma(s)^{r_1} \Gamma(2s)^{r_2}, &\quad \text{where} \\ \mathbf{1} = (1, \ldots, 1) \quad &\text{is the unit element in } \mathbf{C},\ s \in \mathbb{C} \\ r_1 = \text{number of real conjugation classes in } G& \\ r_2 = \text{number of complex conjugation classes in } G& \end{aligned} \tag{3.5}$$

And

$$L_K(s) = \pi^{-\frac{ns}{2}} \Gamma_K(\frac{s}{2}) \tag{3.6}$$

with n the total number of embeddings of K into $\mathbb{C}$ ($n = r_1 + 2r_2$).

The properties of the real gamma function are translated to properties of the generalized gamma function and the functions we have just defined. I'll state only the ones that we will need later:

Proposition 3.6. *The real gamma function satisfies:*

1. $L_K(1) = \frac{1}{\pi^{r_2}}$
2. L_K *is nowhere zero and has a simple pole at* $s = 0$
3. $L_K(s) = A(s) L_K(1-s)$ *, where*

$$A(s) = |d_K|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} L_{\mathbb{C}}(s)^n$$

3.4 Theta Series

Definition 3.7 (Theta Series). For a complete lattice Γ of $\mathbf{R}$, the theta series associated to it is defined by:

$$\theta_\Gamma(z) = \sum_{g \in \Gamma} e^{\pi i \langle gz, g \rangle} \quad \text{for } z \in \mathbf{H}.$$

Theorem 3.8. (Transformation formula for the theta series)

$$\theta_\Gamma(-\frac{1}{z}) = \frac{\sqrt{\mathrm{N}(z/i)}}{\mathrm{vol}(\Gamma)} \theta_{\Gamma'}(z)$$

where Γ' *is the lattice dual to* Γ*, defined by:*

$$\Gamma' = \{g' \in \mathbf{R} \mid \langle g, g' \rangle \in \mathbb{Z} \, \forall g \in \Gamma\}$$

The proof of this transformation formula is a technical one. It is obtained from the Poisson summation formula relating the sum of a function f over a complete lattice Γ in $\mathbf{R}$ and the sum of its Fourier transform $\hat{f}$ over the lattice dual to Γ. The formula is:

$$\sum_{g \text{ in } \Gamma} f(g) = \frac{1}{\mathrm{vol}(\Gamma)} \sum_{g' \in \Gamma'} \hat{f}(g') \tag{3.7}$$

where $\mathrm{vol}(\Gamma)$ is the volume of a fundamental mesh of Γ, and the Fourier transform of f is defined by:

$$\hat{f}(y) = \int_{\mathbf{R}} f(x) e^{-2\pi i \langle x,y \rangle dx}$$

To obtain the theta transformation formula Neukirch actually obtains a more general one for the functions $\theta_p(a,b,z)$, defined as follows (in the next definition, an element $p \in \prod_\tau \mathbb{Z}$ is called admissible if all its components are either zero or one, and at least one component in every complex conjugation class is zero):

Definition 3.9. For $a, b \in \mathbf{R}$ and any admissible $p \in \prod_\tau \mathbb{Z}$ we define:

$$\theta_\Gamma^p(a,b,z) = \sum_{g \in \Gamma} \mathrm{N}((a+g)^p) e^{\pi i \langle (a+g)z, a+g \rangle + 2\pi i \langle b,g \rangle}$$

This series converges absolutely and uniformly on compact subsets of $\mathbf{R} \times \mathbf{R} \times \mathbf{H}$ (and, in particular, the series $\theta_\Gamma(z)$ converges absolutely and uniformly in compact subsets of $\mathbf{H}$, as it is a particular case of this one when $a = b = p = 0$). To obtain theorem (3.8) as a particular case of the more general formula

$$\theta_\Gamma^p(a,b,-\frac{1}{z}) = \left(i^{\mathrm{Tr}(p)} e^{2\pi i \langle a,b \rangle} \operatorname{vol}(\Gamma)\right)^{-1} \mathrm{N}\left((\frac{z}{i})^{p+\frac{1}{2}}\right) \theta_{\Gamma'}^p(-b,a,z) \tag{3.8}$$

we apply Poisson summation formula (3.7) to a function that closely resembles the terms that are added in the definition for $\theta_\Gamma^p(a,b,z)$. This part of the proof will be omitted here, as it would take up too much space. It can again be found in Neukirch [2].

3.5 The Dedekind Zeta Function

Recall the definition we gave in 2.1:

Definition 3.10. The Dedekind zeta function of a number field K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

with the sum running over all the integral ideals of K.

And the theorem we want to prove, taken from [2]:

Theorem 3.11. ***(Analytic continuation of the Dedekind zeta function)***

1. *The Dedekind zeta function $\zeta_K(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$.*

2. *At $s = 1$ it has a simple pole with residue*

$$\operatorname{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_K|}} hR$$

3. It satisfies the functional equation

$$\zeta_K(1-s) = B(s)\zeta_K(s)$$

where

$$B(s) = |d_K|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} L_{\mathbb{C}}(s)^n$$

3.5.1 Step 2

We will follow now the steps mentioned in section 3.2. Step 1 is just the definition of the zeta function, so we start with step 2.

First, we write the series for ζ_K as a summation over the elements of the different ideal classes.

$$\zeta_K(s) = \sum_{\mathfrak{K}} \zeta(\mathfrak{K}, s)$$

$$\zeta(\mathfrak{K}, s) = \sum_{\substack{\mathfrak{a}\in\mathfrak{K} \\ \text{integral}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

The $\zeta(\mathfrak{K}, s)$ are called *partial zeta functions.* Also, we use the following bijection to enumerate more explicitly the integral ideals in a certain ideal class. Take any ideal class $\mathfrak{K}$. Its inverse contains some integral ideal, so we can fix $\mathfrak{a}$ integral such that $\mathfrak{K}$ is the class of $\mathfrak{a}^{-1}$. Then,

$$\begin{aligned} \mathfrak{a}^*/\mathfrak{o}^* &\xrightarrow{\cong} \{\mathfrak{b} \in \mathfrak{K} \mid \mathfrak{b}\,\text{integral}\} \\ \overline{a} &\mapsto \mathfrak{b} = a\mathfrak{a}^{-1} \end{aligned} \tag{3.9}$$

It is not difficult to see that this is indeed a bijection: $a\mathfrak{a}^{-1}$ is integral by definition of the inverse of $\mathfrak{a}$, and two different a, a' give the same $\mathfrak{b}$ only when $(a) = (a')$, so the map is injective. Surjectivity follows from the fact that for any $\mathfrak{b} \in \mathfrak{K}$, $\mathfrak{a}\mathfrak{b}$ is principal and contained in $\mathfrak{a}$.

This allows us to write the partial zeta functions as:

$$\zeta(\mathfrak{K}, s) = \mathfrak{N}(\mathfrak{a})^s \sum_{\overline{a}\in\mathfrak{a}^*/\mathfrak{o}^*} \frac{1}{|\mathrm{N}(\overline{a})|^s}$$

Now we will relate this expression to an integral using the gamma function Γ_K defined in 3.4. In the expression for Γ,

$$\Gamma_K(s) = \int_{\mathbf{R}_+^*} \mathrm{N}(e^{-y}y^s)\frac{dy}{y}$$

substitute in the integral $y \mapsto \pi\,|a|^2\,(y/d_{\mathfrak{a}}^{1/n})$, where a is any complex number, $|\quad|$ denotes the complex norm and $d_{\mathfrak{a}}$ is the absolute value of the discriminant of $\mathfrak{a}$. We obtain:

$$|d_K|^s\,\pi^{-ns}\Gamma_K(s)\frac{\mathfrak{N}(\mathfrak{a})^{2s}}{|\mathrm{N}(a)|^{2s}} = \int_{\mathbf{R}_+^*} e^{-\pi\left\langle ay/d_{\mathfrak{a}}^{1/n},a\right\rangle}\,\mathrm{N}(y)^s\frac{dy}{y}$$

Then sum with a running over a complete system of representatives $\mathfrak{R}$ of $\mathfrak{a}^*/\mathfrak{o}^*$ and get:

$$|d_K|^s \pi^{-ns}\Gamma_K(s)\zeta(\mathfrak{K}, 2s) = \int_{\mathbf{R}_+^*} g(y)\, \mathrm{N}(y)^s \frac{dy}{y}$$

with

$$g(y) = \sum_{a\in\mathfrak{R}} e^{-\pi\left\langle ay/d_{\mathfrak{a}}^{1/n}, a\right\rangle}$$

The function

$$Z_\infty(s) = |d_K|^{s/2} \pi^{-ns/2}\Gamma_K(s/2) = |d_K|^{s/2} L_K(s)$$

is called the *Euler factor at infinity* of the zeta function $\zeta(\mathfrak{K}, s)$ (recall the definition of L_K in (3.6)), and then the *completed zeta function* is written as

$$Z(\mathfrak{K}, s) = Z_\infty(s)\zeta(\mathfrak{K}, s)$$

The sum for g runs over $\mathfrak{R}$, while to relate g to the theta series from definition (3.7) it would be necessary that it run over $\mathfrak{a}$, which can be viewed as a complete lattice in Minkowski space. To write it in this way, break $\mathbf{R}_+^*$ as $\mathbf{S}\times\mathbb{R}_+^*$ with $\mathbf{S} = \left\{x\in\mathbf{R}_+^* \mid \mathrm{N}(x) = 1\right\}$, and break $\mathbf{S}$ further into

$$\mathbf{S} = \bigcup_{\eta\in\mathfrak{o}^*} \eta^2 F$$

where F is a fundamental domain for the action of the group

$$|\mathfrak{o}^*|^2 = \left\{|\epsilon|^2 \mid \epsilon\in\mathfrak{o}^*\right\}$$

on $\mathbf{S}$. The measure d^*x is defined by being the only one such that

$$\frac{dy}{y} = d^*x \times \frac{dt}{t}$$

when $\mathbf{R}_+^*$ is decomposed as the product $\mathbf{S}\times\mathbb{R}_+^*$ (the other measures denote the canonical Haar measures on the respective spaces). It can then be seen that:

$$Z(\mathfrak{K}, 2s) = \frac{1}{w}\int_0^\infty \left(\int_F \theta_{\mathfrak{a}}(ixt^{1/n})d^*x - \mathrm{vol}(F)\right) t^s dt \tag{3.10}$$

where w is the number of roots of unity in K. This w arises as the number of elements in the kernel of $\mathfrak{o}^* \longrightarrow |\mathfrak{o}^*|$, which was proved to be the number of roots of unity when we proved Dirichlet's units theorem. Expression (3.10) does have the form of a Mellin transform,

$$Z(\mathfrak{K}, 2s) = L(f, s) \quad \text{for } f(t) := \frac{1}{w}\int_F \theta_{\mathfrak{a}}(ixt^{1/n})d^*x$$

as we will see later.

3.5.2 Step 3

To apply theorem (3.2) we still need some details. The first one is to calculate $vol(F)$, which can be seen to be equal to $2^{r-1}R$, where R is the regulator of the number field K. The second one is to find the functional equation satisfied by f in order to be able to use it when finding the functional equation for the completed zeta function itself. For this, write $f(t) \equiv f_F(\mathfrak{a}, t)$ to include explicitly the dependence on F and $\mathfrak{a}$ and use the functional equation (3.8). The result is:

$$f_F(\mathfrak{a}, \frac{1}{t}) = t^{1/2} f_{F^{-1}}((\mathfrak{a}\mathfrak{D})^{-1}, t)$$
$$f_F(\mathfrak{a}, t) = \frac{2^{r-1}}{w} R + O(e^{-ct^{1/n}}) \quad \text{for } t \to \infty,\ c > 0$$

where $\mathfrak{D}$ is the different of $K \mid \mathbb{Q}$, that appears here because $(\mathfrak{a}\mathfrak{D})^{-1}$ is the lattice dual to $\mathfrak{a}$ when ideals are regarded as lattices in Minkowski space.

3.5.3 Step 4

Finally, the resulting functional equation for the partial completed zeta functions is:

$$Z(\mathfrak{K}, s) = Z(\mathfrak{K}', 1-s) \quad \text{where } \mathfrak{K}\mathfrak{K}' = [\mathfrak{D}] \quad \text{, the ideal class of } \mathfrak{D} \tag{3.11}$$

And the values of its residues in 0 and 1 are, respectively

$$-\frac{2^r}{w} R \quad \text{and} \quad \frac{2^r}{w} R$$

An immediate consequence of this is the functional equation for the completed zeta function (obtained adding the partial ones), which has exactly the same form. The residues are h times those of the partial functions, for h the class number of K, as we sum over the h classes to get the completed zeta function.

3.5.4 Step 5

From this we can deduce the functional equation for the Dedekind zeta function easily as follows:

We have that $\zeta_K(s) = Z_\infty(s)^{-1} Z_K(s)$. As $Z_\infty(s) = |d_K|^{s/2} L_X(s)$ we see that Z_∞ is only zero at $s = 0$, so it cancels the pole of $Z_K(s)$ at this point. Hence, $\zeta_K(s)$ can be extended to the complex plane with an only pole at $s = 1$ as can be seen from this expression, and its residue at $s = 1$ is the residue of $Z_K(s)$ times $Z_\infty(1)^{-1}$ (as $Z_\infty(s)$ has no pole at $s = 1$). This gives us the formula for the residue at $s = 1$:

$$\mathrm{Res}_{s=1}\, \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_K|}} hR$$

which is called the *analytic class number formula.*

For the functional equation of ζ_K, note that using 3.6 we have an equation for $Z_\infty(s)$:

$$\begin{aligned} Z_\infty(1-s) &= |d_K|^{\frac{1-s}{2}} L_K(1-s) \\ &= |d_K|^{\frac{s}{2}} |d_K|^{\frac{1}{2}-s} A(s)^{-1} L_K(s) \\ &= |d_K|^{\frac{1}{2}-s} A(s)^{-1} Z_\infty(s) \end{aligned} \tag{3.12}$$

So ζ_K satisfies:

$$\begin{aligned} \zeta_K(1-s) &= Z_\infty(1-s)^{-1} Z_K(1-s) \\ &= |d_K|^{s-\frac{1}{2}} A(s) Z_\infty(s)^{-1} Z_K(s) \\ &= |d_K|^{s-\frac{1}{2}} A(s) \zeta_K(s) \end{aligned} \tag{3.13}$$

In short,

$$\zeta_K(1-s) = B(s)\zeta_K(s)$$

with

$$B(s) = |d_K|^{s-\frac{1}{2}} \left(\cos\frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin\frac{\pi s}{2}\right)^{r_2} L_{\mathbb{C}}(s)^n$$

We have thus proved theorem (3.11).

References

[1] Harvey Cohn, *Advanced number theory*, Dover Publications, Inc., 1998.

[2] J urgen Neukirch, *Algebraic number theory*, Springer, 1999.

[3] Eric Weisstein, *http://mathworld.wolfram.com/classnumber.html*.