# Relative $p$-adic Hodge Theory

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$K :=$ a perfect field of characteristic $p$.
$G_K := \mathrm{Gal}(\overline{K}/K)$.

For a finite dimensional $\mathbb{F}_p$-vector space $V$ equipped with a continuous $G_K$-action, we define

$$D(V) = (V \otimes_{\mathbb{F}_p} \overline{K})^{G_K}.$$

**Theorem (Lang)**

*The functor $V \mapsto D(V)$ gives a rank preserving equivalence of categories from the category of discrete representations of $G_K$ on finite dimensional $\mathbb{F}_p$-vector spaces to the category of $\varphi$-modules over $K$; a quasi inverse functor is given by $D \mapsto V(D) = (D \otimes_K \overline{K})^{\varphi=1}$.*

For any ring $S$ equipped with an endomorphism $\varphi$, a $\varphi$-module over $S$ is a finite free $S$-module $M$ equipped with a semilinear $\varphi$-action such that $M$ is isomorphic to its $\varphi$-pullback as an $S$-module.

In the characteristic $p$ case, the endomorphism $\varphi$ is the $p$-th power Frobenius. In the mixed characteristic case, $\varphi$ is a lift of the $p$-th power Frobenius.

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For any finite free $\mathbb{Z}_p$-module $V$ equipped with a continuous $G_K$-action, we define

$$D(V) = (V \otimes_{\mathbb{Z}_p} W(\overline{K}))^{G_K}.$$

### Corollary

*The functor $V \mapsto D(V)$ gives a rank preserving equivalence of categories from the category of continuous representations of $G_K$ on finite free $\mathbb{Z}_p$-modules to the category of $\varphi$-modules over $W(K)$; a quasi inverse functor is given by $D \mapsto V(D) = (D \otimes_{W(K)} W(\overline{K}))^{\varphi=1}$.*

Now we further equip $K$ with a complete multiplicative nonarchimedean norm $|\cdot|$.

### Definition

For $r > 0$, define $\widetilde{\mathcal{R}}_K^{\mathrm{int},r}$ the ring of *overconvergent Witt vectors of radius $r$* to be the set of $f = \sum_{i=0}^{\infty} p^i [x_i] \in W(K)$ for which $\lim_{i\to\infty} p^{-i}|x_i|^r = 0$. We define the norm $|\cdot|_r$ on $\widetilde{\mathcal{R}}_K^{\mathrm{int},r}$ by setting $|f|_r = \max_{i\in\mathbb{N}}\{p^{-i}|x_i|^r\}$. Then $\widetilde{\mathcal{R}}_K^{\mathrm{int},r}$ is complete with respect to $|\cdot|_r$. Let $\widetilde{\mathcal{R}}_K^{\mathrm{int}} = \cup_{r>0}\widetilde{\mathcal{R}}_K^{\mathrm{int},r}$.

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For any finite free $\mathbb{Z}_p$-module $V$ equipped with a continuous $G_K$-action, we define

$$D^\dagger(V) = (V \otimes_{\mathbb{Z}_p} \widetilde{\mathcal{R}}^{\mathrm{int}}_K)^{G_K}.$$

## Theorem

*The functor $V \mapsto D^\dagger(V)$ gives a rank preserving equivalence of categories from continuous representations of $G_K$ on finite free $\mathbb{Z}_p$-modules to the category of $\varphi$-modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}_K$; a quasi inverse functor is given by $D^\dagger \mapsto V(D^\dagger) = (D^\dagger \otimes_{\widetilde{\mathcal{R}}^{\mathrm{int}}_K} \widetilde{\mathcal{R}}^{\mathrm{int}}_K)^{\varphi=1}$.*

Let $\widetilde{\mathcal{R}}^{\mathrm{bd},r}_K = \widetilde{\mathcal{R}}^{\mathrm{int},r}_K[1/p]$. Let $\widetilde{\mathcal{R}}^r_K$ be the Fréchet completion of $\widetilde{\mathcal{R}}^{\mathrm{bd},r}_K$ under the norms $|\cdot|_s$ for $s \in (0, r]$.

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**Theorem**

*The following categories are equivalent.*

1. *The category of continuous representations of $G_K$ on finite dimensional $\mathbb{Q}_p$-vector spaces.*
2. *The category of étale $\varphi$-modules over $\widetilde{\mathcal{E}}_K$.*
3. *The category of étale $\varphi$-modules over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$.*
4. *The category of étale $\varphi$-modules over $\widetilde{\mathcal{R}}_K$.*

*More precisely, the functors from* (3) *to* (2) *and from* (3) *to* (4) *are base extensions.*

For a $p$-adic representation we mean a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of the the absolute Galois group of (mixed characteristic) local fields with perfect residue field. Fontaine's theory of $\varphi, \Gamma$-modules classifies $p$-adic representations into various type of *$(\varphi, \Gamma)$-modules* which we will explain as below.

We restrict to $G_{\mathbb{Q}_p}$-representations for simplicity. Let $H = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\mu_{p^\infty}))$, and let $\Gamma = G_{\mathbb{Q}_p}/H = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. The starting point of Fontaine's theory is the following

**Theorem (Fontaine-Wintenberger)**

$\mathrm{Gal}(\mathbb{F}_p((\bar{\pi}))^{\mathrm{sep}}/\mathbb{F}_p((\bar{\pi}))) \cong H$

Define

$$\mathcal{O}_{\mathcal{E}} = \varprojlim_{n\to\infty} \frac{\mathbb{Z}_p[[\pi]][\pi^{-1}]}{(p^n)} = \left\{ \sum_{i\in\mathbb{Z}} a_i\pi^i : a_i \in \mathbb{Z}_p, \lim_{i\to-\infty} |a_i|_p = 0 \right\}.$$

which is a cohen ring with residue field $\mathbb{F}_p((\bar{\pi}))$. Let $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$.

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Define

$$\varphi(f(\pi)) = f((1+\pi)^p - 1), \qquad g(f(\pi)) = f((1+\pi)^{\chi(g)} - 1), \quad g \in G_{\mathbb{Q}_p}.$$

where $\chi$ is the $p$-adic cyclotomic character.

For any $p$-adic representation $V$ we define

$$D(V) = (V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}^{\mathrm{ur}}})^H$$

which is a $(\varphi, \Gamma)$-module over $\mathcal{E}$. For a $(\varphi, \Gamma)$-*module* we mean a $\varphi$-module equipped with a continuous $\Gamma$-action which commutes with $\varphi$.

The $p$-adic completion of the direct limit

$$\mathcal{O}_{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{E}} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{E}} \to \cdots$$

is isomorphic to $W(\mathbb{F}_p((\bar{\pi}))^{\mathrm{perf}})$. In this way, we identify $\mathcal{O}_{\mathcal{E}}$ as a subring of $W(\mathbb{F}_p((\bar{\pi}))^{\mathrm{perf}})$.

We equip $\mathbb{F}_p((\bar{\pi}))$ with a multiplicative norm $|\cdot|$ by setting $|\bar{\pi}| = p^{-\frac{p-1}{p}}$. This norm extends naturally to $\mathbb{F}_p((\bar{\pi}))^{\mathrm{perf}}$. Let

$$\mathcal{R}^{\mathrm{int},r} = \mathcal{O}_{calE} \cap \widetilde{\mathcal{R}}^{\mathrm{int},r}_{\widehat{\mathbb{F}_p((\bar{\pi}))^{\mathrm{perf}}}} \quad \mathcal{R}^{\mathrm{bd},r} = \mathcal{E} \cap \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{\widehat{\mathbb{F}_p((\bar{\pi}))^{\mathrm{perf}}}}.$$

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More precisely,

$$\mathcal{R}^{\mathrm{bd},r} = \left\{ \sum_{i \in \mathbb{Z}} a_i \pi^i : a_i \in \mathbb{Q}_p, \sup_i \{|a_i|_p\} < +\infty, \lim_{i \to \pm\infty} p^{-ir} |a_i|_p = 0, s \in (0, r] \right.$$

Let $\mathcal{R}^r$ be the Fréchet completion of $\mathcal{R}^{\mathrm{bd},r}$ under the norms $|\cdot|_s$ for $s \in (0, r]$. It follows that

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Let $\mathcal{R}^{\mathrm{int}} = \cup_{r>0}\mathcal{R}^{\mathrm{int},r}, \mathcal{R}^{\mathrm{bd}} = \cup_{r>0}\mathcal{R}^{\mathrm{bd},r}, \mathcal{R} = \cup_{r>0}\mathcal{R}^r$. A $(\varphi, \Gamma)$-module over $\mathcal{E}$ is called étale if it is the base change of a $(\varphi, \Gamma)$-module over $\mathcal{O}_{\mathcal{E}}$. A $(\varphi, \Gamma)$-module over $\mathcal{R}^{\mathrm{bd}}$ or $\mathcal{R}$ is called étale if it is the base change of a $(\varphi, \Gamma)$-module over $\mathcal{R}^{\mathrm{int}}$.

Theorem (Fontaine, Cherbonnier-Colmez, Berger, Kedlaya)

*The following categories are equivalent.*

1. *The category of continuous representations of* $G_{\mathbb{Q}_p}$ *on finite dimensional* $\mathbb{Q}_p$*-vector spaces.*
2. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathcal{E}$*.*
3. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathcal{R}^{\mathrm{bd}}$*.*
4. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathcal{R}$*.*

*More precisely, the functors from* (3) *to* (2) *and from* (3) *to* (4) *are base extensions.*

Goal: generalize Fontaine's theory of $(\varphi, \Gamma)$-modules to local systems over nonarchimedean analytic spaces over mixed characteristic local fields with perfect residue fields.

Step 1: non-abelian Artin-Schreier theory for general bases.
Step 2: "norm rings" for affinoid spaces.
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From now on, we fix a perfect Banach algebra $R$ over an analytic field of characteristic $p$. We denote its norm by $|\cdot|$.

For $r > 0$, let $\widetilde{\mathcal{R}}_R^{\text{int},r}$ be the ring of $f = \sum_{i=0}^{\infty} p^i[x_i] \in W(R)$ for which $\lim_{i\to\infty} p^{-i}|x_i|^r = 0$, and define $|\cdot|_r$ on $\widetilde{\mathcal{R}}_R^{\text{int},r}$ by setting

$$|f|_r = \max_{i\in\mathbb{N}}\{p^{-i}|x_i|^r\}.$$

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## Definition

A *$\varphi$-module* over $W(R)$ (resp. $\widetilde{\mathcal{E}}_R$, $\widetilde{\mathcal{R}}_R^{\text{int}}$, $\widetilde{\mathcal{R}}_R^{\text{bd}}$) is a finite locally free module $M$ equipped with an isomorphism $\varphi^* M \cong M$. A *$\varphi$-module* over $\widetilde{\mathcal{R}}_R$ is a vector bundle $M$ over $\widetilde{\mathcal{R}}_R^r$ for some $r > 0$, together with an isomorphism $\varphi^* M \cong M$ of vector bundles over $\widetilde{\mathcal{R}}_R^s$ for some $s \in (0, r/p]$.

## Theorem (Kedlaya-L.)

*The following categories are equivalent.*

(1) *The category of étale $\mathbb{Q}_p$-local systems over* $\mathrm{Spec}(R)$.

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### Remark

*For a $\varphi$-module $M$ over $\widetilde{\mathcal{R}}_R$, the point $\alpha \in \mathcal{M}(R)$ so that $M_\alpha$ is étale forms an open subset of $\mathcal{M}(R)$. Furthermore, $M$ is étale if and only if $M_\alpha$ is étale for any $\alpha \in \mathcal{M}(R)$.*

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### Definition

Let $S$ be perfect ring of characteristic $p$ equipped with the trivial norm, and we equip $W(S)$ with the $p$-adic norm. We define continuous maps $\lambda : \mathcal{M}(S) \to \mathcal{M}(W(S))$ and $\mu : \mathcal{M}(W(S)) \to \mathcal{M}(S)$ as follows. For $\alpha \in \mathcal{M}(S)$, we define $\lambda(\alpha)$ by setting

$$\lambda(\alpha)\left(\sum_{i=0}^{\infty} p^i[\overline{x}_i]\right) = \sup_i\{p^{-i}\alpha(\overline{x}_i)\}.$$

For $\beta \in \mathcal{M}(W(S))$, we define $\mu(\beta)$ by setting

$$\mu(\beta)(\overline{x}) = \beta([\overline{x}]).$$

It is easy to check that $\mu \circ \lambda = \text{id}$ and $\lambda \circ \mu(\beta) \geq \beta$. In fact, one can construct a homotopy between $\lambda \circ \mu$ and the identity map on $\mathcal{M}(W(R))$, and show that any subset of $\mathcal{M}(R)$ has the same homotopy type as its inverse image of $\mu$. Intuitively, one can view that $\mu : \mathcal{M}(W(S)) \to \mathcal{M}(S)$ realize $\mathcal{M}(W(S))$ as a disk bundle over $\mathcal{M}(S)$ and $\lambda$ is a section of it.

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An element $z = \sum_{i=0}^{\infty} p^i[\overline{z}_i] \in W(\mathfrak{o}_R)$ is called *primitive of degree* 1 if $\overline{z}_0 \in R^\times$, $\alpha(\overline{z}_0) = p^{-1}$, $\alpha(\overline{z}_0^{-1}) = p$, and $\overline{z}_1 \in \mathfrak{o}_R^\times$.

**Theorem (Kedlaya)**

*Let $\mathfrak{o}_R = \{a \in R \,|\, |a| \le 1\}$. Equip $W(\mathfrak{o}_R)$ with the norm $\lambda(|\cdot|)$. Suppose that $z \in W(\mathfrak{o}_R)$ is primitive of degree 1.*

(a) *For each $\gamma \in \mathcal{M}(\mathfrak{o}_R)$, the quotient seminorm $\sigma(\gamma)$ on $W(\mathfrak{o}_R)/(z)$ induced by $\lambda(\gamma)$ is multiplicative and satisfies $\mu(\sigma(\gamma)) = \gamma$.*

(b) *Then the map $\sigma : \mathcal{M}(\mathfrak{o}_R) \to \mathcal{M}(W(\mathfrak{o}_R))$ indicated by (a) is a continuous section of $\mu$, which induces a homeomorphism of $\mathcal{M}(\mathfrak{o}_R)$ with $\mathcal{M}(W(\mathfrak{o}_R)/(z))$. Under this homeomorphism, Laurent (resp. rational) subspaces of $\mathcal{M}(\mathfrak{o}_R)$ correspond to Laurent (resp. rational) subspaces of $\mathcal{M}(W(\mathfrak{o}_R)/(z))$.*

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We fix a "$p$-adic $\exp(2\pi i)$" $\epsilon = (1, \epsilon_1, \epsilon_2, \cdots)$. The minimal polynomial of $\epsilon_n$ is

$$F_n = \frac{(1+\pi)^{p^n} - 1}{(1+\pi)^{p^{n-1}} - 1}.$$

Note that $\mathcal{R}^{\mathrm{int},1}/(F_n) \cong \mathbb{Q}_p(\epsilon_n)$ and $\varphi(F_n) = F_{n+1}$. The $\varphi$-action thus induces the following diagram

$$\begin{array}{ccccc}
\mathcal{R}^{\mathrm{int},1}/(F_1) & \xrightarrow{\varphi} & \mathcal{R}^{\mathrm{int},1}/(F_2) & \xrightarrow{\varphi} & \cdots \\
\downarrow \cong & & \downarrow \cong & & \\
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\end{array}$$

where the maps of the second row are inclusions.

Taking $p$-adic completion of the direct limit of this diagram we get

$$\widetilde{\mathcal{R}}^{\mathrm{int},1}_{\widehat{\mathbb{F}_p((\pi))^{\mathrm{perf}}}}/(z) \cong \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$$

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$$z = \varphi^{-1}(F_1) = \frac{\pi}{\pi^{1/p}} = \sum_{i=0}^{p-1} [1+\overline{\pi}]^{i/p}$$

We fix a "$p$-adic $\exp(2\pi i)$" $\epsilon = (1, \epsilon_1, \epsilon_2, \cdots)$. The minimal polynomial of $\epsilon_n$ is

$$F_n = \frac{(1+\pi)^{p^n} - 1}{(1+\pi)^{p^{n-1}} - 1}.$$

Note that $\mathcal{R}^{\mathrm{int},1}/(F_n) \cong \mathbb{Q}_p(\epsilon_n)$ and $\varphi(F_n) = F_{n+1}$. The $\varphi$-action thus induces the following diagram

$$\begin{array}{ccccc} \mathcal{R}^{\mathrm{int},1}/(F_1) & \xrightarrow{\varphi} & \mathcal{R}^{\mathrm{int},1}/(F_2) & \xrightarrow{\varphi} & \cdots \\ \downarrow \cong & & \downarrow \cong & & \\ \mathbb{Q}_p(\epsilon_1) & \longrightarrow & \mathbb{Q}_p(\epsilon_2) & \longrightarrow & \cdots \end{array}$$

where the maps of the second row are inclusions.

Taking $p$-adic completion of the direct limit of this diagram we get

$$\widetilde{\mathcal{R}}^{\mathrm{int},1}_{\widehat{\mathbb{F}_p((\pi))^{\mathrm{perf}}}}/(z) \cong \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$$

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which is primitive of degree 1.

For any $S$, let $\mathbf{F\acute{E}t}(S)$ denote the category of finite étale algebras over $S$.

**Proposition (Kedlaya-L.)**

*Suppose that $L$ is a perfect analytic field of characteristic $p$, and that $z \in W(\mathfrak{o}_L)$ is primitive of degree 1. Applying $\mathbf{F\acute{E}t}$ to any arrow in the diagram*

$$L \leftarrow \tilde{\mathcal{R}}_L^{\mathrm{int},1} \to \tilde{\mathcal{R}}_L^{\mathrm{int},1}/(z)$$

*produces a rank-preserving equivalence of tensor categories.*

For $L = \widehat{\mathbb{F}_p((\pi))}^{\mathrm{perf}}$, this proposition recovers Fontaine-Wintenberger's theorem.

**Theorem (Kedlaya-L.)**

*For $z \in W(\mathfrak{o}_R)$ primitive of degree 1, applying $\mathbf{F\acute{E}t}$ to any arrow in the diagram*

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### Definition

For $J$ a finite index set, let $\mathbb{Q}_p\langle J\rangle$ denote the completion for the Gauss norm of the polynomial ring $\mathbb{Q}_p[J]$. That is, for each labeling $j_1, \dots, j_m$ of the elements of $J$, $\mathbb{Q}_p\langle J\rangle$ is isomorphic to $\mathbb{Q}_p\langle T_1, \dots, T_m\rangle$, but we do not distinguish a choice of labeling. By a *framed* $\mathbb{Q}_p$*-affinoid algebra*, we will mean a pair $(A, \psi)$ in which $A = A(\psi)$ is a reduced $\mathbb{Q}_p$-affinoid algebra and $\psi : \mathbb{Q}_p\langle J\rangle \to A$ is a bound homomorphism for some finite set $J = J(\psi)$ which identifies $\mathcal{M}(A)$ with a closed immersed subspace of a rational subspace of $\mathcal{M}(\mathbb{Q}_p\langle J\rangle)$. In this setting, we refer to $\mathcal{M}(A)$ as a *framed* $\mathbb{Q}_p$*-affinoid space*.

### Definition

A *morphism* $(A_1, \psi_1) \to (A_2, \psi_2)$ of framed $\mathbb{Q}_p$-affinoid algebras is a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_p\langle J(\psi_1)\rangle & \xrightarrow{\psi_1} & A_1 \\ \downarrow & & \downarrow \\ \mathbb{Q}_p\langle J(\psi_2)\rangle & \xrightarrow{\psi_2} & A_2 \end{array} \tag{14.1}$$

in which the vertical arrows are bound homomorphisms, and the left vertical arrow is induced by a function $J(\tau) : J(\psi_1) \to J(\psi_2)$. A *basic morphism* $(A_1, \psi_1) \to (A_2, \psi_2)$ is a morphism in which $J(\tau)$ is injective. A basic morphism whose underlying homomorphism $A_1 \to A_2$ is an isomorphism is called a *basic refinement*; in this situation, we say that $(A_2, \psi_2)$ is a *basic refinement* of $(A_1, \psi_1)$.

## Definition

Let $(A_1, \psi_1)$, $(A_2, \psi_2)$ be framed $\mathbb{Q}_p$-affinoid algebras. Let $\tau : A_1 \to A_2$ be a bounded homomorphism of $\mathbb{Q}_p$-affinoid algebras. We define the *framed graph* of $\tau$ to be the framed $\mathbb{Q}_p$-affinoid algebra $(A_3, \psi_3)$ in which $A_3 = A_2$, $J(\psi_3) = J(\psi_1) \cup J(\psi_2)$, and $\psi_3 : \mathbb{Q}_p\langle J(\psi_3)\rangle \to A_3$ is given by identifying $\mathbb{Q}_p\langle J(\psi_1) \cup J(\psi_2)\rangle$ with $\mathbb{Q}_p\langle J(\psi_1)\rangle \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p\langle J(\psi_2)\rangle$ and taking the product of the morphisms $\psi_1$ and $\psi_2 \circ \tau$. By including the two factors into the completed tensor product, we obtain morphisms $(A_1, \psi_1) \to (A_3, \psi_3)$ and $(A_2, \psi_2) \to (A_3, \psi_3)$ whose underlying morphisms are the map $\tau : A_1 \to A_2 \cong A_3$ and the identification $A_2 \cong A_3$.

## Remark

*Using framed graphs, we see that if we start with the category of framed $\mathbb{Q}_p$-affinoid algebras with basic morphisms, then localize by formally inverting the basic refinements, we recover the category of $\mathbb{Q}_p$-affinoid algebras.*

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### Definition

Equip $\mathbb{Z}_p\langle J\rangle[\![\pi]\!]$ with the Gauss norm. Define the Frobenius lift $\varphi$ on $\mathbb{Z}_p\langle J\rangle[\![\pi]\!]$ by the formula

$$\sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}\pi^{i_0}T_1^{i_1}\cdots T_m^{i_m} \mapsto \sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}((1+\pi)^p-1)^{i_0}T_1^{pi_1}\cdots T_m^{pi_m}.$$

Define an action of the group $\Gamma_J = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^J$ commuting with $\varphi$, by declaring that $\gamma \in \mathbb{Z}_p^\times$ acts via the formula

$$\sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}\pi^{i_0}T_1^{i_1}\cdots T_m^{i_m} \mapsto \sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}((1+\pi)^\gamma-1)^{i_0}T_1^{i_1}\cdots T_m^{i_m},$$

while $(e_1,\dots,e_m)\in\mathbb{Z}_p^J$ acts via the formula

$$\sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}\pi^{i_0}T_1^{i_1}\cdots T_m^{i_m} \mapsto \sum_{i_0,\dots,i_m} c_{i_0,\dots,i_m}(1+\pi)^{e_1i_1+\cdots+e_mi_m}\pi^{i_0}T_1^{i_1}\cdots T_m^{i_m}.$$

Note that

$$\mathbb{Q}_p\langle J\rangle[\![\pi]\!]/(F_n) \cong \mathbb{Q}_p(\epsilon_n)\langle J\rangle.$$

The Frobenius lift $\varphi$ induces the isometric homomorphism

$$\varphi_n : \mathbb{Q}_p(\epsilon_n)\langle J\rangle \to \mathbb{Q}_p(\epsilon_{n+1})\langle J\rangle$$

which is the identity on scalars, and maps $T_i$ to $T_i^p$.

**Proposition (Kedlaya-L.)**

*The map*

$$\varphi_n^* : \mathcal{M}(\mathbb{Q}_p(\epsilon_{n+1})\langle J\rangle) \mapsto \mathcal{M}(\mathbb{Q}_p(\epsilon_n)\langle J\rangle)$$

*is surjective, with finite fibres permuted transitively by* $(\mathbb{Z}_p^\times \cap p^n\mathbb{Z}_p) \ltimes p^n\mathbb{Z}_p^J$.

Let $(A, \psi)$ be a framed $\mathbb{Q}_p$-affinoid algebra with $J(\psi) = J$. Let $A_{\psi,n}$ be the reduced quotient of

$$A\widehat{\otimes}_{\mathbb{Q}_p\langle J\rangle,\varphi_{n-1}\circ\cdots\circ\varphi_0}\mathbb{Q}_p(\epsilon_n)\langle J\rangle.$$

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Equip $A_{\psi,n}$ with the spectral seminorm, then we may identify $\mathcal{M}(A_{\psi,n})$ with the inverse image of $\mathcal{M}(A) \subseteq \mathcal{M}(\mathbb{Q}_p\langle J\rangle)$ in $\mathcal{M}(\mathbb{Q}_p(\epsilon_n)\langle J\rangle)$; it is closed and $\Gamma_J$-stable. We have

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \mathcal{M}(A_{\psi,1}) & \longrightarrow & \mathcal{M}(A_{\psi,0}) \\
 & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{\varphi_1^*} & \mathcal{M}(\mathbb{Q}_p(\epsilon_1)\langle J\rangle) & \xrightarrow{\varphi_0^*} & \mathcal{M}(\mathbb{Q}_p\langle J\rangle).
\end{array}$$

Let $A_{\psi,\infty}$ be the completion of the direct limit

$$A_{\psi,0} \stackrel{\varphi_0}{\to} A_{\psi,1} \stackrel{\varphi_1}{\to} \cdots ,$$

which carries an action of $\Gamma_J$ with fixed subring $A_{\psi,0} \cong A$.
Let $\mathfrak{o}_{\overline{A_\psi}}$ be the inverse limit of

$$\cdots \stackrel{\varphi}{\to} \mathfrak{o}_{A_{\psi,\infty}}/(p) \stackrel{\varphi}{\to} \mathfrak{o}_{A_{\psi,\infty}}/(p),$$

and let $\overline{A_\psi} = \mathfrak{o}_{\overline{A_\psi}}[\overline{\pi}^{-1}]$.

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Equip $A_{\psi,n}$ with the spectral seminorm, then we may identify $\mathcal{M}(A_{\psi,n})$ with the inverse image of $\mathcal{M}(A) \subseteq \mathcal{M}(\mathbb{Q}_p\langle J\rangle)$ in $\mathcal{M}(\mathbb{Q}_p(\epsilon_n)\langle J\rangle)$; it is closed and $\Gamma_J$-stable. We have

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and let $\overline{A_\psi} = \mathfrak{o}_{\overline{A_\psi}}[\overline{\pi}^{-1}]$.

**Proposition**

*$A_{\psi,\infty}$ is canonically isomorphic to $\tilde{\mathcal{R}}^{\mathrm{int},1}_{A_\psi}/(z)$.*

We take $A_\psi$ to be the "norm ring" for $(A, \psi)$.

### Definition

$$\tilde{\mathbf{B}}_\psi = W(\overline{A}_\psi)[p^{-1}], \qquad \tilde{\mathbf{B}}^{\dagger,r}_\psi = \tilde{\mathcal{R}}^{\mathrm{bd},r}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{B}}^{\dagger}_\psi = \tilde{\mathcal{R}}^{\mathrm{bd}}_{\overline{A}_\psi}$$
$$\tilde{\mathbf{C}}^{[s,r]}_\psi = \tilde{\mathcal{R}}^{[s,r]}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{C}}^{r}_\psi = \tilde{\mathcal{R}}^{r}_{\overline{A}_\psi}, \qquad \tilde{\mathbf{C}}_\psi = \tilde{\mathcal{R}}_{\overline{A}_\psi}.$$

For $0 < s \leq r$, let $\mathbf{C}^{[s,r]}_\psi$ be the completion of $\mathfrak{o}_A[\![\pi]\!]$ in $\tilde{\mathbf{C}}^{[s,r]}_\psi$ under the norm $\max\{\lambda(\overline{\beta}^r_\psi), \lambda(\overline{\beta}^s_\psi)\}$. Then put

$$\mathbf{C}^r_\psi = \cap_{0<s\leq r}\mathbf{C}^{[s,r]}_\psi, \quad \mathbf{C}_\psi = \cup_{r>0}\mathbf{C}^r_\psi,$$
$$\mathbf{B}^{\dagger,r}_\psi = \tilde{\mathbf{B}}^{\dagger}_\psi \cap \mathbf{C}^r_\psi, \qquad \mathbf{B}^{\dagger}_\psi = \cup_{r>0}\mathbf{B}^r_\psi.$$

Let $\mathbf{B}_\psi$ be the $p$-adic completion of $\mathbf{B}^\dagger_\psi$.

### Theorem (Kedlaya-L.)

*The following categories are equivalent.*

1. *The category of étale* $\mathbb{Q}_p$*-local systems over* $\mathcal{M}(\mathcal{A})$.
2. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathbf{B}_{\psi}$.
3. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathbf{B}^{\dagger}_{\psi}$.
4. *The category of étale* $(\varphi, \Gamma)$*-modules over* $\mathbf{C}_{\psi}$.

*More precisely, the functors from* (3) *to* (2) *and from* (3) *to* (4) *are base extensions.*

K.S. Kedlaya and R. Liu, Relative *p*-adic Hodge theory *I*,
http://math.mit.edu/ kedlaya/papers/.