Motivic Integrals of K3 surfaces over Non-Archimedean Fields

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P-adic Integral

Let X be a smooth complete variety over a non-archimedean local field K with the ring of integers $R \subset K$ and the residue field k of order q.

A top degree differential form $\omega \in \Gamma(X, \Omega_X^{\dim X})$ defines a finite real valued measure $|\omega|$ on the set of K-points of X. By base change, for every finite extension $K \subset K'$, we get a number $\int_{X(K')} |\omega|$.

Definition. A weak Néron model of X is a smooth scheme $\mathcal{V}$ over R whose generic fiber is X and such that every point of X with values in an unramified extension $K' \supset K$ extends to a R'-point of $\mathcal{V}$.

Example. If $\overline{X}$ is a proper regular model of X over R, then $\mathcal{V} := \overline{X} - X_{sing}$ is a weak Néron model.

A weak Néron model always exists but it is not unique.

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From now on we assume that the canonical bundle $\Omega_X^{dim\,X}$ is trivial.

If $\mathcal{V}$ is a weak Néron model of X, we have

$$\int_{X(K)} |\omega| = \sum_i |V_i^\circ(k)| q^{-r_i}, \tag{0.1}$$

where V_i° are the connected components of the special fiber of $\mathcal{V}$ and $r_i \in \mathbb{Z}$ are defined by $div(\omega) = \sum_i r_i [V_i^\circ]$.

In particular, the quantity at the right-hand side of the equation (0.1) does not depend on the choice of $\mathcal{V}$ (but does depend on ω).

The renormalized integral

$$\int_{X(K)} := \sum_i [V_i^\circ(k)] q^{-r_i + \min_i r_i}$$

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If X has a smooth and proper model over R the Grothendieck-Lefschetz formula together with the Proper Base Change and Local Acyclicity theorems yield a cohomological interpretation for the normalized measure:

$$\int_{X(K)} = \sum_j (-1)^j Tr(F^{-1}, H^j(X_{\overline{K}}, \mathbb{Q}_l)), \tag{0.2}$$

where l is prime number different from the characteristic of k and $F \in Gal(\overline{K}/K)$ is a lifting of the Frobenius automorphism $Fr \in Gal(\overline{k}/k)$.

Problem. Find a generalization of the formula (0.2) to the case of bad reduction.

Theorem

If X admits a proper strictly semi-stable model over R then the formula (0.2) holds modulo $q-1$.

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Proof. Let $\overline{X}$ be a strictly semi-stable model of X. Then $\int_{X(K)} \equiv |Y_{sm}(k)|(mod(q-1))$.

On the other hand, by the Grothendieck-Lefschetz formula

$$\sum_j (-1)^j Tr(F^{-1}, H^j(X_{\overline{K}}, \mathbb{Q}_l)) = \sum_j (-1)^j Tr(F^{-1}, H^j(Y_{\overline{k}}, \Psi(\mathbb{Q}_l))) =$$

$$\sum_{y \in Y(k)} \sum_i (-1)^i Tr(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_l))_y).$$

If $y \in Y_{sm}(k)$ the corresponding sum equals 1.

If $y \in Y_{sing}(k)$ then $\mathcal{H}^i(\Psi(\mathbb{Q}_l))_y \simeq \bigwedge^i T(-i)$, where T is a vector space with the trivial action of $Gal(\overline{K}/K)$. Thus,

$$\sum_i (-1)^i Tr(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_l))_y) \equiv \sum_i (-1)^i dim \bigwedge^i T \equiv 0(mod(q-1)).$$

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Motivic Integral.

Definition. Let k be a field. The Grothendieck group of varieties over k, $K_0(Var_k)$, is the free group on the isomorphism classes of varieties modulo the relation $[Y] = [Z] + [Y \backslash Z]$, where $Z \subset Y$ is a closed subvariety. In addition to being a group, $K_0(Var_k)$ has a ring structure given by $[Y \times_k Z] = [Y] \cdot [Z]$. Let $\mathbb{Z}(-i)$ represent the class of $\mathbb{A}^i$ in $K_0(Var_k)$ and denote the localization $K_0(Var_k)[\mathbb{Z}(-1)^{-1}]$ by $K_0(Var_k)_{loc}$.

Notation: $\mathbb{Z}(i) := \mathbb{Z}(-i)^{-1}$, for $i > 0$. $M(i) := M \cdot \mathbb{Z}(i)$, for $i \in \mathbb{Z}$, $M \in K_0(Var_k)_{loc}$.

Theorem (Kontsevich, Loeser-Sebag)

Let K be a non-archimedean field with a perfect residue field k, and let X be a smooth Calabi-Yau variety over K. Then the element (called the motivic integral) $\int_X := \sum_i [V_i^\circ](r_i - \min_i r_i) \in K_0(Var_k)_{loc}$, is independent of the choice of a weak Néron model $\mathcal{V}$.

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Motivic Integral.

If $k = \mathbb{F}_q$, we recover the normalized volume by taking the image of the motivic integral under the homomorphism

$$K_0(Var_{\mathbb{F}_q})_{loc} \to \mathbb{Z}_{(q)} \quad [Z] \rightsquigarrow |Z(\mathbb{F}_q)|.$$

Let

$$R^{Hodge} : K_0(Var_{\mathbb{C}})_{loc} \to K_0(MHS)$$

be the homomorphism from the Grothendieck ring of varieties to the Grothendieck ring of mixed $\mathbb{Q}$-Hodge structures that takes the class of a variety Z to $\sum(-1)^i[H^i_c(Z)]$. The Hodge integral $R^{Hodge}(\int_{\mathcal{X}})$ is the image of the motivic integral under the above morphism of Grothendieck rings.

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The limit Hodge structure of a variety over $\mathbb{C}((t))$.

Building upon the Schmid-Steenbrink construction, with every smooth projective variety X over $\mathbb{C}((t))$ we associate a mixed Hodge structure $H^m(\lim X)$ equipped with the monodromy action, called the limit Hodge structure. A rough idea: Steenbrink attached a mixed Hodge structure to every normal crossing log scheme over the log point. Applying his construction to the special fiber Y of a strictly semi-stable model $\overline{X}$ of X over $R = \mathbb{C}[[t]]$ we get our $H^m(\lim X)$. We prove the independence of the choice of a model and the functoriality.

Theorem

For every strictly semi-stable degeneration of Calabi-Yau varieties the image of $R^{Hodge}(\int_X)$ in the quotient ring $K_0(MHS)/(\mathbb{Q}(1)-\mathbb{Q})$ equals the alternated sum of the classes of the limit Hodge structures:

$$R^{Hodge}\left(\int_X\right) = \sum (-1)^i [H^i(\lim X)] \quad mod(\mathbb{Q}(1)-\mathbb{Q}).$$

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K3 surfaces over $\mathbb{C}((t))$.

Let X be a smooth projective K3 surface over $\mathbb{C}((t))$ and let

$$H^2(\lim X) = (H^2(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}} \subset H^2(\lim X, \mathbb{Q}), F_i \subset H^2(\lim X, \mathbb{C}))$$

be the corresponding limit Hodge structure. Assume that the monodromy acts on $H^2(\lim X, \mathbb{Z})$ by a unipotent operator. Then, its logarithm is known to be integral:

$$N : H^2(\lim X, \mathbb{Z}) \to H^2(\lim X, \mathbb{Z}).$$

Set $W_i^{\mathbb{Z}} = W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$. The morphisms

$$Gr\, N^1 : W_3^{\mathbb{Z}} / W_2^{\mathbb{Z}} \to W_1^{\mathbb{Z}} / W_0^{\mathbb{Z}}$$

$$Gr\, N^2 : W_4^{\mathbb{Z}} / W_3^{\mathbb{Z}} \to W_0^{\mathbb{Z}}$$

are injective and have finite cokernels. Let r_1, r_2 be their orders.

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Theorem

Let X be a smooth projective K3 surfaces over $\mathbb{C}((t))$, $X_e = X \otimes_{\mathbb{C}((t))} \mathbb{C}((\sqrt[e]{t}))$. Assume that X has a strictly semi-stable model over $\mathbb{C}[[t]]$. Then $N^3 = 0$.

(a) *If $N^2 = 0$ then*

$$\int_{X_e} = 2\mathbb{Z}(0) - (e\sqrt{r_1}+1)[E] + 20\mathbb{Z}(-1) + (e\sqrt{r_1}-1)[E](-1) + 2\mathbb{Z}(-2),$$

where E is the elliptic curve defined by the rank 2 Hodge structure on $W_1^{\mathbb{Z}} = W_1^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$.

(b) *If $N^2 \neq 0$ then*

$$\int_{X_e} = \left(\frac{e^2 r_2}{2} + 2\right)\mathbb{Z}(0) + (20 - e^2 r_2)\mathbb{Z}(-1) + \left(\frac{e^2 r_2}{2} + 2\right)\mathbb{Z}(-2).$$

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$$\int_{X_e} = 2\mathbb{Z}(0) - (e\sqrt{r_1}+1)[E] + 20\mathbb{Z}(-1) + (e\sqrt{r_1}-1)[E](-1) + 2\mathbb{Z}(-2),$$

where E is the elliptic curve defined by the rank 2 Hodge structure on $W_1^{\mathbb{Z}} = W_1^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$.

(b) *If $N^2 \neq 0$ then*

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K3 surfaces over $\mathbb{C}((t))$.

Theorem

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Proof.

Let us explain the idea of our proof assuming that $e = 1$. Let $\overline{C}$ be a smooth curve over $\mathbb{C}$, $a \in \overline{C}$ a point, $C = \overline{C} - a$. First, using the theory of Hilbert schemes and Artin's approximation theorem, we reduce the proof to the case when X is extended to a smooth proper scheme $\mathcal{X}$ over C.

The rest of the proof is based on a result of Kulikov asserting the existence of a strictly semi-stable model $\overline{\mathcal{X}} \xrightarrow{\overline{\pi}} \overline{C}$ such that the canonical bundle $\omega_{\overline{\mathcal{X}}}$ is trivial over an open neighborhood of the special fiber Y. For any such model, we have

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Moreover, the special fiber Y has a very special form. If $N^2 \neq 0$ the Clemens polytope $Cl(Y)$ of Y is a triangulation of a sphere, all the irreducible components of Y are smooth rational surfaces and all the double curves are rational.

It follows that

$$\int_X = c_1\mathbb{Z}(0) + c_2\mathbb{Z}(-1) + c_3\mathbb{Z}(-2),$$

for some $c_i \in \mathbb{Z}$.

Friedman and Scattone proved that the canonical morphism

$$H^2(Cl(Y), \mathbb{Z}) \xrightarrow{\sim} H^2_{Zar}(Y, \mathbb{Z}) \to W_0^{\mathbb{Z}}$$

is an isomorphism. Let

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For every $x \in H^2(\lim X, \mathbb{Z})$, we have the Picard-Lefschetz formula

$$< x, N^2(x) >= \sum_j a_j^2,$$

where $\phi(x) = \sum a_j \delta_j$ and δ_j are the 2-simplecies of $Cl(Y)$, $a_j \in \mathbb{Z}$.

Pick x such that $\phi(x)$ is a generator of $H_2(Cl(Y), \mathbb{Z})$. Because, $Cl(Y)$ is a manifold, we get

$$\phi(x) = \sum \pm\delta_j$$

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The proof is completed using the $mod(\mathbb{Q} - \mathbb{Q}(1))$Theorem.

If $N^2 = 0$ the proof is similar but simpler.

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Monodromy pairing

Let X be a smooth scheme over a non-archimedean field K, $X^{an}_{\widehat{\overline{K}}}$ the analytic space associated with the scheme $X \otimes_K \widehat{\overline{K}}$ over the completion of an algebraic closure $\overline{K}$, and let $|X^{an}_{\widehat{\overline{K}}}|$ be the underlying topological space.

We denote by $\Gamma^m_\Lambda(X)$ the singular cohomology of the space $|X^{an}_{\widehat{\overline{K}}}|$ with coefficients in a commutative ring Λ. Equivalently, the group $\Gamma^m_\Lambda(X)$ can be defined as the cohomology of rigid analytic space associated with $X \otimes_K \widehat{\overline{K}}$ with coefficients in the constant sheaf Λ for the Tate G-topology.

It is proven by Berkovich, Hrushovsky and Loeser that $\Gamma^m_\Lambda(X)$ is a finitely generated Λ-module and that it is a birational invariant of (smooth) X.

If $\overline{X}$ is a proper strictly semi-stable scheme over R generic and special fibers X, Y, we have

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Monodromy pairing

Theorem

If char k $\neq$ l, we have

$$\Gamma^m_{\mathbb{Q}_l}(X) \xrightarrow{\sim} Im(H^m(X_{\overline{K}}, \mathbb{Q}_l)(m) \xrightarrow{N^m} H^m(X_{\overline{K}}, \mathbb{Q}_l)),$$

where N is the logarithm of the monodromy operator.

A different description of the space $\Gamma^m_{\mathbb{Q}_l}(X)$ in the case of finite residue field was obtained earlier by Berkovich.

Monodromy pairing

Let X be a smooth proper variety of dimension d. Given $x, y \in N^d H^d(X_{\overline{K}}, \mathbb{Q}_l)$ we set

$$(x, y)_l = (-1)^{\frac{d(d-1)}{2}} < x, y' >,$$

where $y' \in H^d(X_{\overline{K}}, \mathbb{Q}_l)$ is an element such that $N^d y' = y$ and $<, >$ is the Poincaré pairing.

Theorem

The restriction of $(x, y)_l$ to $\Gamma^m_{\mathbb{Q}}(X)$ defines a positive pairing (the monodromy pairing)

$$(\cdot, \cdot) : \Gamma^d_{\mathbb{Q}}(X) \otimes \Gamma^d_{\mathbb{Q}}(X) \to \mathbb{Q} \tag{0.3}$$

The pairing (0.3) is independent of l and it is a birational invariant of X.

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Monodromy pairing

We define a numeric (birational) invariant of X to be

$$r_d(X, K) = \frac{1}{Disc(\cdot, \cdot)}.$$

The formula

We shall say that a smooth projective d-dimensional Calabi-Yau variety X over K is maximally degenerated if $\Gamma^d(X) \otimes \mathbb{Q} \neq 0$.

Conjecture

Let X be a smooth projective maximally degenerated K3 surface over K. Then There exists a finite extension $K' \supset K$ such that for every finite extension $L \supset K'$ of ramification index e

$$\int_{X \otimes L} = \left(\frac{e^2 r_2}{2} + 2\right) \mathbb{Q}(0) + (20 - e^2 r_2)\mathbb{Q}(-1) + \left(\frac{e^2 r_2}{2} + 2\right) \mathbb{Q}(-2).$$

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The Conjecture is true if X is a Kummer K3 surface and char $k \neq 2$.

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