The arithmetic curve, Witt vectors, and zeta

jt work with C. Consani

- The counting function and the adèle class space.
- The BC-system, Witt vectors and p-adic representations.
- First hints of the arithmetic curve.

Hasse-Weil zeta function

For an irreducible, smooth and projective algebraic curve X over a finite field $\mathbb{F}_q$, the Hasse-Weil zeta function is of the form

$$Z(q,T) = \exp\left(\sum_{r\geq 1} N(q^r)T^r/r\right)$$

where the counting function $N(q^r)$ is the number $\#X(\mathbb{F}_{q^r})$ of points over $\mathbb{F}_{q^r}$.

Soulé's limit definition for $q \to 1$

$N(q)$ given, and defined for $q \in [1, \infty)$

$$Z(q,T) = \exp\left(\sum_{r \geq 1} N(q^r)T^r/r\right)$$

$$\zeta_N(s) = \lim_{q \to 1} (q-1)^{\chi}\, Z(q, q^{-s})\,, \quad \chi = N(1)$$

Problem :

Can one find $N(q)$ such that $\zeta_N(s)$ is the complete Riemann zeta function :

$$\zeta_N(s) = \zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Two difficulties

$N(1) = -\infty$ does not allow one to use the limit definition and seems to contradict the positivity of $N(q)$.

Lemma :

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) u^{-s} d^*u$$

From the lemma one gets

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = -\int_1^\infty N(u)\, u^{-s} d^*u \,.$$

Theorem (CC)

(1) The counting function $N(q)$ satisfying the above requirements exists as a distribution and is given by the formula

$$N(q) = q - \frac{d}{dq}\left(\sum_{\rho \in Z} \text{order}(\rho) \frac{q^{\rho+1}}{\rho+1}\right) + 1$$

where Z is the set of non-trivial zeros of the Riemann zeta function and the derivative is taken in the sense of *distributions*.

(2) The function $N(q)$ is positive (as a *distribution*) for $q > 1$.

(3) The value $N(1)$ is equal to $-\infty$ and reflects precisely the distribution of the zeros of zeta in $E \log E$.

According to Soulé the value $N(1)$ of the counting function is the Euler characteristic of the hypothetical "curve" C over $\mathbb{F}_1$. Since C has infinite genus one thus gets a priori that

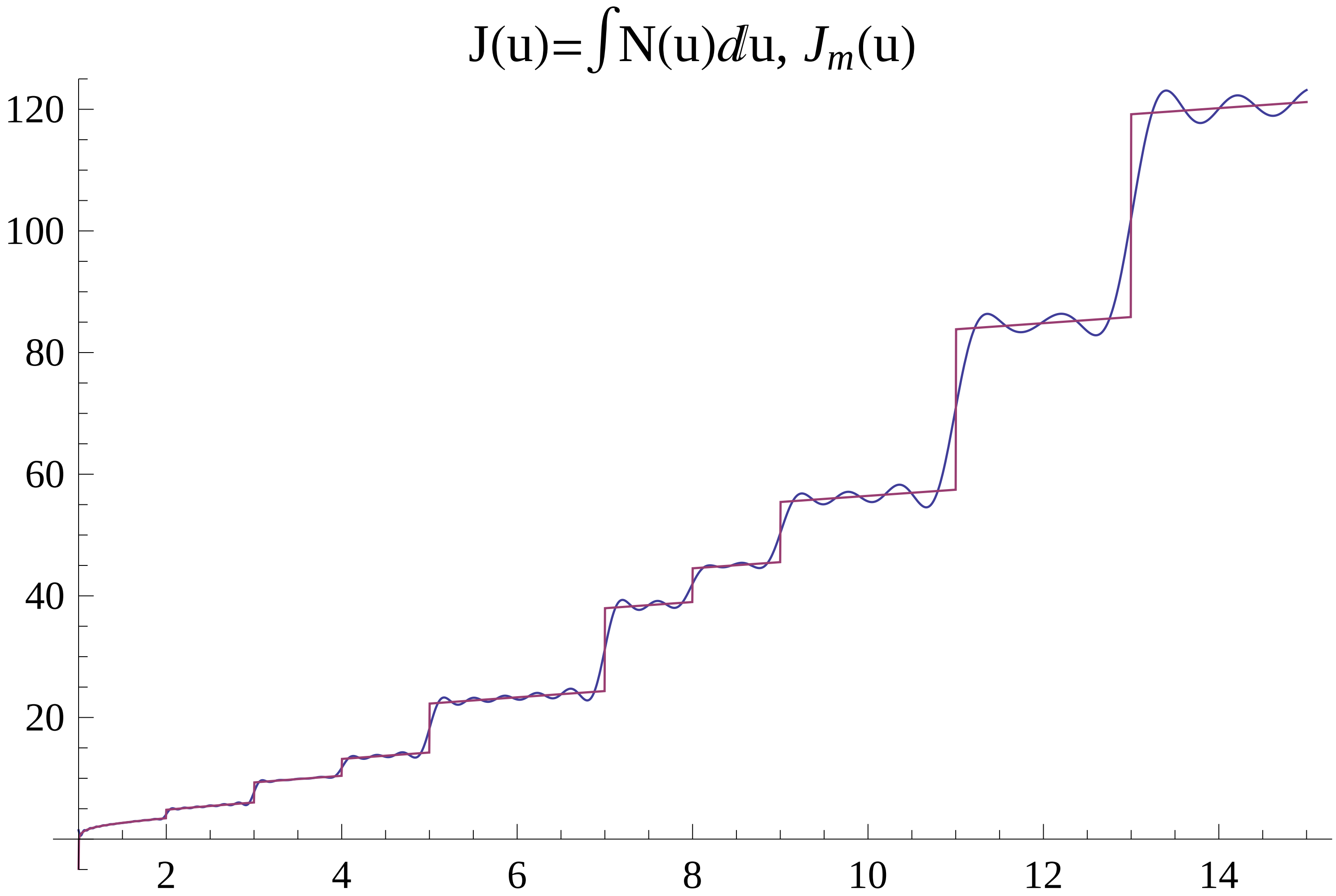
$$N(1) = -\infty$$

hence creating a tension with the expected positivity of $N(q)$ for $q > 1$. This tension is resolved by the Theorem, since the distribution $N(q)$ is positive for $q > 1$ but its value at $q = 1$ is given formally by

$$N(1) = 2 - \lim_{\epsilon \to 0} \frac{\omega(1+\epsilon) - \omega(1)}{\epsilon}$$

$$\sim -\frac{1}{2} E \log E \,, \;\; E = \frac{1}{\epsilon}$$

whose behavior when $\epsilon \to 0$ even reflects the density of zeros.


$J(u)=\int N(u)\,du,\ J_m(u)$
120
100
80
60
40
20
2
4
6
8
10
12
14

Curve $C = \overline{\text{Spec}\,\mathbb{Z}}$?

While the counting function associated to an algebraic variety defined over a finite field takes only integer values this is no more the case for the counting function $N(q)$ described above. In the function field case the counting function $N(q)$ counts the number of rational points of the associated curve fixed under the action of the corresponding power of the arithmetic Frobenius and for this reason one finds that $N(q^k) \leq N(q^\ell)$ when k divides ℓ. This is no longer the case for the distribution $N(u)$ which has a contribution of the form

$$\kappa(u) = \frac{u^2}{u^2 - 1}$$

Explicit Formulas (Riemann)

One lets

$$\widehat{g}(s) = \int_1^\infty g(u)\, u^s d^* u$$

The explicit formula then takes the form

$$\widehat{g}(0) + \widehat{g}(1) - \sum_{\rho \in Z} \text{order}(\rho) \widehat{g}(\rho)$$

$$= \sum_p \sum_{m=1}^{\infty} \log p \; g(p^m) +$$

$$\left(\frac{\gamma}{2} + \frac{\log \pi}{2}\right) g(1) + \int_1^\infty \frac{t^2 g(t) - g(1)}{t^2 - 1} d^* t$$

Let us apply this formula with the function g_x determined by

$$g_x(u) = u \quad \text{for } u \in [1, x]\,, \quad g_x(u) = 0 \text{ for } u > x$$

The left-hand side of the explicit formula gives, up to a constant

$$J(x) = \frac{x^2}{2} + x - \sum_{\rho \in Z} \text{order}(\rho) \frac{x^{1+\rho}}{1+\rho}$$

Thus the left-hand side of the explicit formula gives a natural primitive $J(x)$ of the counting function $N(x)$.

It is thus natural to differentiate formally the family g_x with respect to x and see what the right-hand side of the explicit formula looks like. One gets

$$\partial_x g_x = u\delta(u-x)$$

It is characterized as a distribution by

$$\int b(u) g_x(u) d^*u = b(x)$$

This equality means that the distributional trace of $\int g_x(u)\vartheta_u d^*u$ is the distributional trace of ϑ_x since

$$\int g_x(u)\vartheta_u d^*u = \vartheta_x$$

Thus we can express $N(u)$ as an intersection number provided we can interpret the explicit formulas as trace formulas.

Explicit Formulas (Weil)

Let $\mathbb{K}$ be a global field, α a nontrivial character of $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$ and $\alpha = \prod \alpha_v$ its local factors. Let $h \in \mathcal{S}(C_{\mathbb{K}})$ have compact support. Then

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\chi \in \widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \widehat{h}(\tilde{\chi}, \rho) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u$$

where $\int'$ is normalized by α_v and

$$\widehat{h}(\chi, z) = \int h(u)\, \chi(u)\, |u|^z \, d^*u.$$

Fixed points of $C_{\mathbb{K}}$ on $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

Let $(x,u) \in M \times C_{\mathbb{K}} \,|\, u\,x = x$, with $u \neq 1$. There exists $v \in \Sigma_{\mathbb{K}}$ with

$$x \in \mathfrak{p}_v = \{x \in M \,|\, x_v = 0\}$$

The isotropy group of any $x \in \mathfrak{p}_v$ contains

$$\mathbb{K}_v^* \subset C_{\mathbb{K}}\,, \quad \mathbb{K}_v^* = \{(k_w) \,|\, k_w = 1 \ \forall w \neq v\}$$

Trace of the action of $\mathbb{K}_v^*$ on $\mathbb{K}_v$

$$(T\,\xi)\,(x) = \int k(x,y)\,\xi(y)\,dy$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) = \int k(x,x)\,dx\,.$$

$$T\xi(x) = \xi(\lambda x)\,, k(x,y) = \delta(\lambda x - y)$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) = \int k(x,x)\,dx = \int \delta(\lambda x - x)\,dx$$

$$= \frac{1}{|\lambda - 1|}\int \delta(z)\,dz = \frac{1}{|\lambda - 1|}$$

(with $z = (\lambda - 1)x$)

Trace of the action of $C_{\mathbb{K}}$ on $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

The transverse space to $\mathfrak{p}_v \subset M$ is the local field $\mathbb{K}_v$. The isotropy group $I_v = \mathbb{K}_v^*$ acts as $\mathbb{K}_v^*$ on $\mathbb{K}_v$ by multiplication

$$\vartheta(u)\xi(x) = \xi(u^{-1}x)$$

$$\mathrm{Tr}_{\mathrm{distr}}\left(\int h(u)\vartheta(u)d^*u\right) = \sum_v \int_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u$$

Algebraic structure of $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

At first $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ was used as a (noncommutative) space but the following structures gradually emerged :

– $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ is a monoid, and one can apply the geometry of monoids going back to Kato $\rightarrow$ spectral realization on $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$.
– $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ is a hyperring over the Krasner hyperfield $\mathbf{K}$, this gives $\Pi_1^{\mathrm{ab}}(X)' \sim P(\mathbb{H}_{\mathbb{K}})$ in positive characteristic.

Spectral realization and $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$

Theorem (CC) : The cohomology $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ gives the spectral realization of zeros of L-functions. The spectrum of the action of $C_{\mathbb{K}}$ on $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ by $\vartheta[-\frac{1}{2}]$ is invariant by the symmetry $\chi(g) \mapsto \chi(g^{-1})$ of Grössencharakters of $\mathbb{K}$.

The sheaf $\Omega[-\frac{1}{2}]$ is that of half-densities :

$$f(x)|dx|^{\frac{1}{2}} \sim f(x)|x|^{\frac{1}{2}}|d^*x|^{\frac{1}{2}}$$

The BC-system as a Hecke algebra

One considers the inclusion $P_{\mathbb{Z}}^{+} \subset P_{\mathbb{Q}}^{+}$ where the "$ax+b$" algebraic group P is viewed as the functor which to any abelian ring R assigns the group P_R of 2 by 2 matrices over R of the form

$$P_R = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \ ; \ a, b \in R \, , \ a \text{ invertible} \right\} .$$

Here $\Gamma_0 = P_{\mathbb{Z}}^{+}$ and $\Gamma = P_{\mathbb{Q}}^{+}$ denote the restrictions to $a > 0$. This inclusion $\Gamma_0 \subset \Gamma$ is such that the orbits of the left action of Γ_0 on Γ/Γ_0 are all *finite*. The same clearly holds for orbits of Γ_0 acting on the right on $\Gamma_0 \backslash \Gamma$.

Hecke algebra $\mathcal{H}_{\mathbb{Q}}$

The Hecke algebra $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is by definition the convolution algebra of functions of finite support

$$f : \Gamma_0 \backslash \Gamma \to \mathbb{Q},$$

which fulfill the Γ_0-invariance condition

$$f(\gamma\gamma_0) = f(\gamma), \qquad \forall \gamma \in \Gamma, \forall \gamma_0 \in \Gamma_0,$$

The convolution product is then defined by the formula

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma\gamma_1^{-1}) f_2(\gamma_1)\,.$$

$$\mathcal{H}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}}(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+})$$

It contains the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.

One lets $e(r) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, be the canonical generators for $r \in \mathbb{Q}/\mathbb{Z}$. For each $n \in \mathbb{N}$, one defines endomorphisms σ_n of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ by $\sigma_n(e(\gamma)) = e(n\gamma)$ and additive maps $\tilde{\rho}_n$ by

$$\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]\,, \quad \tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma').$$

Presentation

The integral BC-algebra $\mathcal{H}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$ is the algebra generated by the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, and by the elements $\tilde{\mu}_n$ and μ_n^*, with $n \in \mathbb{N}$, which satisfy the relations :

$$\tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x)$$
$$\mu_n^* x = \sigma_n(x) \mu_n^*$$
$$x \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x),$$

where $\tilde{\rho}_m$, $m \in \mathbb{N}$ as above, as well as the relations

$$\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m \,, \quad \forall n, m$$
$$\mu_{nm}^* = \mu_n^* \mu_m^* \,, \quad \forall n, m$$
$$\mu_n^* \tilde{\mu}_n = n$$
$$\tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n \qquad (n, m) = 1.$$

Time evolution

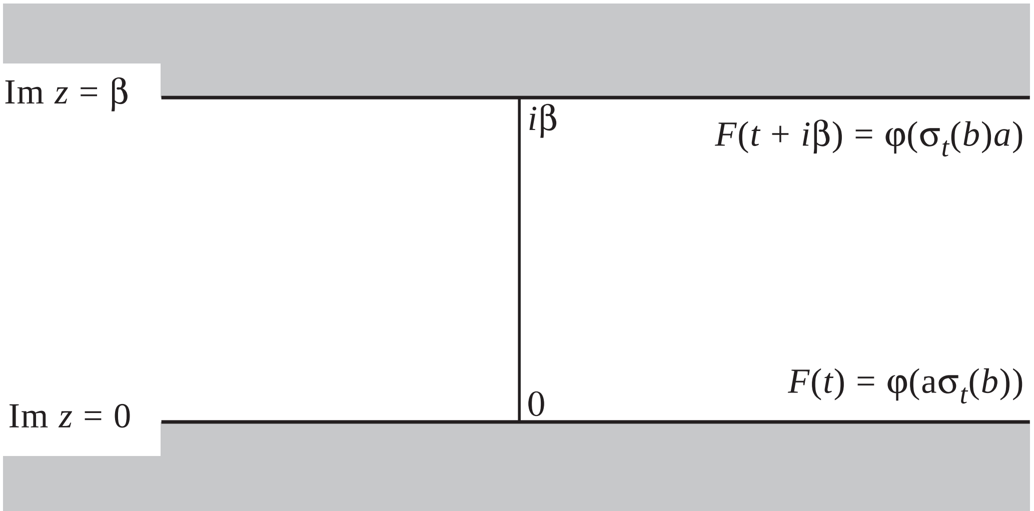
Regular representation of the Hecke algebra $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma/\Gamma_0)$, and canonical time evolution.

The time evolution is given by the following one-parameter group of automorphisms of the algebra $\mathcal{H}_{\mathbb{C}}$:

$$\sigma_t(\tilde{\mu}_n) = n^{it}\tilde{\mu}_n, \quad \sigma_t(\mu_n^*) = n^{-it}\mu_n^*,$$

$$\sigma_t(e(r)) = e(r).$$

KMS Condition



$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \quad F_{x,y}(t+i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Phase transition with SSB

The unique KMS state above critical temperature is

$$\varphi_\beta\,(e(a/b)) = b^{-\beta} \prod_{p \text{ prime},\ p|b} \left(\frac{1-p^{\beta-1}}{1-p^{-1}} \right),$$

and the extremal KMS states below critical temperature are

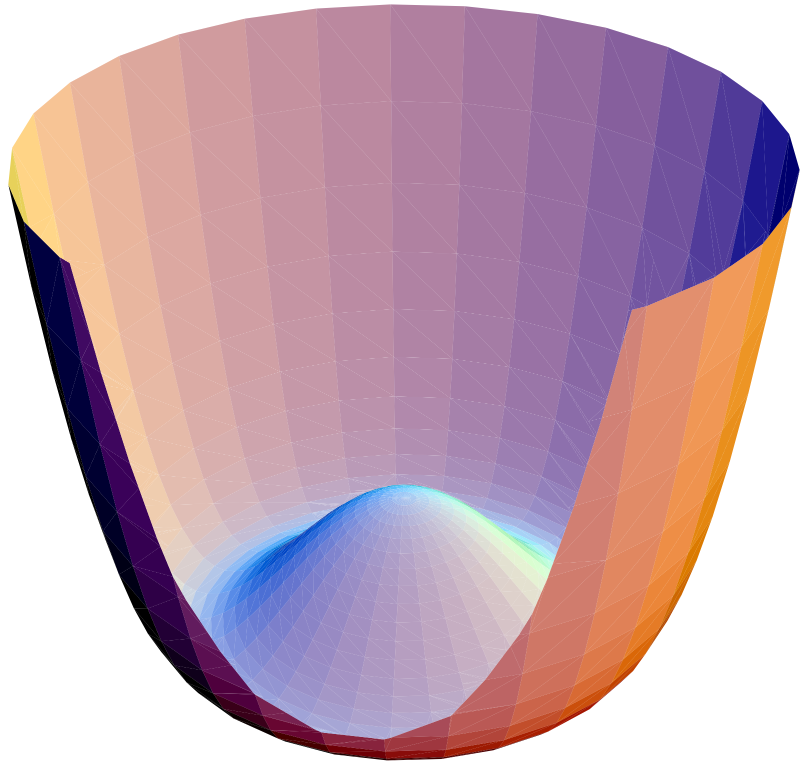
$$\varphi_{\beta,\rho}(e(a/b)) = \frac{\mathrm{Tr}(\pi_\rho(e(a/b))e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

$$= \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta} \rho(\zeta_{a/b}^n),$$

where π_ρ is the representation of the algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ given by

$$\pi_\rho(\mu_n)\epsilon_m = \epsilon_{nm}, \qquad \pi_\rho(e(a/b))\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m,$$

where $\rho \in \hat{\mathbb{Z}}^*$ determines an embedding in $\mathbb{C}$ of the cyclotomic field $\mathbb{Q}^{cycl}$ generated by the abstract roots of unity.



Three Witt functors

$$\mathbb{W}_0(A) = K_0(\underline{\text{End}}_A)/K_0(A).$$

The key additional structures are given by

1. The Teichmüller lift which is a multiplicative map $\tau : A \to \mathbb{W}_0(A)$.
2. The Frobenius endomorphisms F_n for $n \in \mathbb{N}$.
3. The Verschiebung (shift) additive functorial endomorphisms V_n, $n \in \mathbb{N}$.
4. The ghost components $\text{gh}_n : \mathbb{W}_0(A) \to A$ for $n \in \mathbb{N}$.

The BC-system and $\mathbb{W}_0(\bar{\mathbb{F}}_p)$

To each $\sigma : \bar{\mathbb{F}}_p^\times \sim \mu^{(p)} \subset \mu \subset \mathbb{C}^\times$, corresponds a canonical isomorphism $\tilde{\sigma}$

$$\mathbb{W}_0(\bar{\mathbb{F}}_p) \xrightarrow{\tilde{\sigma}} \mathbb{Z}[\mu^{(p)}] \subset \mathbb{Z}[\mu] \xrightarrow{r=id\otimes\epsilon} \mathbb{Z}[\mu^{(p)}]$$

The Frobenius F_n and Verschiebung maps V_n of $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ are obtained by restriction of the endomorphisms σ_n and maps $\tilde{\rho}_n$ of $\mathbb{Z}[\mu] = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ by the formulas

$$\tilde{\sigma} \circ F_n = \sigma_n \circ \tilde{\sigma}\,, \quad \tilde{\sigma} \circ V_n = r \circ \tilde{\rho}_n \circ \tilde{\sigma}$$

This Theorem shows that the integral BC-system with its full structure is, if one drops the p-component, completely described as $\mathbb{W}_0(\bar{\mathbb{F}}_p)$. As a corollary one gets a representation π_σ of the integral BC-system $\mathcal{H}_{\mathbb{Z}}$ on $\mathbb{W}_0(\bar{\mathbb{F}}_p)$,

$$\pi_\sigma(x)\xi = \tilde{\sigma}^{-1}(r(x))\,\xi\,, \quad \pi_\sigma(\mu_n^*) = F_n\,, \quad \pi_\sigma(\tilde{\mu}_n) = V_n$$

for all $\xi \in \mathbb{W}_0(\bar{\mathbb{F}}_p)$, $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $n \in \mathbb{N}$.

$\underline{\mathbb{W}(A)}$

$$L(\tau(f)) = (1 - tf)^{-1} \in \Lambda(A) := 1 + tA[[t]]$$

$$(1 + \sum a_n t^n) \star (1 + \sum b_n t^n) = 1 + a_1 b_1 t +$$

$$+ \left(a_1^2 b_1^2 - a_2 b_1^2 - a_1^2 b_2 + 2 a_2 b_2 \right) t^2 + \ldots$$

$$\varphi_A : \mathbb{W}(A) \to \Lambda(A), \qquad x = (x_n)_{n \in \mathbb{N}} \mapsto f_x(t) = \prod_{\mathbb{N}} (1 - x_n t^n)^{-1}$$

$\mathbb{W}_{p^\infty}(A)$

$\mathbb{W}_{p^\infty}$ is the Witt functor using the set of powers of the prime p.

$$\mathbb{W}(\bar{\mathbb{F}}_p) = (\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p))^{I(p)}$$

where $I(p) \subset \mathbb{N}$ is the set of integers which are prime to p. Let $\overline{\mathbb{Q}_p^{\mathsf{un}}} \subset \mathbb{C}_p$ be the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathsf{un}}$ of p-adic numbers. Then $\mathbb{W}_{p^\infty}(\bar{\mathbb{F}}_p)$ is the completion $\overline{\mathbb{Z}_p^{\mathsf{un}}} \subset \overline{\mathbb{Q}_p^{\mathsf{un}}}$ of the subring generated by roots of unity.

$\mathbb{W}(\bar{\mathbb{F}}_p)$ and p-adic representations of the BC-system

Let $\sigma \in X_p$ and $\rho : \mathbb{Q}^{\mathrm{cycl,p}} \to \mathbb{C}_p$ the associated embedding. The representation π_σ of the integral BC-system $\mathcal{H}_\mathbb{Z}$ extends to $\mathbb{W}(\bar{\mathbb{F}}_p)$. For $n \in I(p)$, the $\pi_\sigma(\mu_n)$ and for $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, the $\pi_\sigma(x)$ are $\overline{\mathbb{Z}_p^{\mathrm{un}}}$-linear operators such that

$$\pi_\sigma(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\sigma(e(a/b))\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m\,, \quad \forall n \in \mathbb{N},\, m, b \in I(p).$$

Moreover

$$\pi_\sigma(\mu_p) = \mathrm{Fr}^{-1}$$

is the inverse of the Frobenius automorphism, acting componentwise as a skew linear operator.

Partition function and p-adic L-functions

$$Z(\frac{a}{b},\beta) = \sum_{m\in I(p)} \rho(\zeta_{a/b}^{m})\, m^{-\beta}$$

$$\langle c\rangle^{p-1} = c^{p-1}\,, \quad \langle c\rangle^{s} := \sum_{0}^{\infty} \binom{s}{j} (\langle c\rangle - 1)^{j}$$

$$Z(\frac{a}{b},\beta) := \frac{1}{bq} \sum_{1\le c<bq,\, c\notin p\mathbb{N}} \rho(\zeta_{a/b}^{c})\, \frac{\langle c\rangle^{1-\beta}}{\beta-1} \sum_{0}^{\infty} \binom{1-\beta}{j} (\frac{bp}{c})^{j} B_{j}$$

The formula

$$\ell_\beta(z) = \sum_{n=1}^{\infty} n^{-\beta} z^n$$

defines the multiple logarithm and fulfills

$$z\partial_z \ell_\beta(z) = \ell_{\beta-1}(z)\,.$$

For $\beta = 0$ the sum gives the rational fraction $\ell_0(z) = \frac{z}{1-z}$ and this shows that when $\beta \in -\mathbb{N}$ is a negative integer $\ell_\beta(z)$ is a rational fraction. Thus it makes sense over any field. For β a negative odd integer of the form $\beta = 1 - m = 1 - k(p-1)$, one has

$$(1 - p^{-\beta}\mathrm{Fr})^{-1} Z(\frac{a}{b}, \beta) = \ell_\beta(\rho(\zeta_{a/b})) \in \mathbb{Q}^{\mathrm{cyc}}$$

which is formally independent of the prime p.

KMS at a prime p

Since when $\beta \in -\mathbb{N}$ is a negative integer $\ell_\beta(z)$ is a rational fraction, one can prove identities for arbitrary $\beta \in D$ by checking them on these special values. For any $m \in \mathbb{N}$ and ζ_m a primitive m-th root of unity, one has

$$\frac{1}{m}\sum_{j=0}^{m-1} \ell_\beta(\zeta_m^j x) = m^{-\beta}\ell_\beta(x^m)$$

One can show the analogue of the KMS$_\beta$ condition for the $\mathbb{C}_p$-valued functionals on the integral BC-system $\mathcal{H}_\mathbb{Z}$ such that, in particular,

$$\varphi_{\beta,\rho}(e(a/b)) = \frac{1}{Z(\beta)} Z_\rho(\frac{a}{b},\beta).$$

First hints towards the "curve" for the global field $\mathbb{Q}$

In the case of number fields, the groupoid of prime elements of the hyperring $\mathbb{H}_{\mathbb{K}}$ still makes sense and the issue is to construct, in characteristic zero, a geometric model similar to the curve and its abelian cover in the function field case. Let $\mathbb{K}$ be a function field, *i.e.* a global field of characteristic $p > 1$ and let $\mathbb{F}_q \subset \mathbb{K}$ be the field of constants. The abelian cover of the curve is obtained by the following steps

(1) One considers the maximal abelian extension $\mathbb{K}^{\mathrm{ab}} \supset \mathbb{K}$ of $\mathbb{K}$.

(2) One considers inside $\mathbb{K}^{\mathsf{ab}}$ the finite extensions $E \supset \bar{\mathbb{F}}_p \otimes_{\mathbb{F}_q} \mathbb{K}$ of $\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_q} \mathbb{K}$.

(3) For each such extension E the space of (discrete) valuations is turned into a scheme with non-empty open sets given by complements of finite subsets and structure sheaf given by the intersection of the valuation rings inside E.

The space $\mathrm{Val}_p(\mathbb{Q}^{\mathrm{cyc}})$

The space $\mathrm{Val}_p(\mathbb{Q}^{\mathrm{cyc}})$, of valuations on $\mathbb{Q}^{\mathrm{cyc}}$ extending the p-adic valuation of $\mathbb{Q}$ is canonically isomorphic to the quotient Σ_p of the space X_p of isomorphisms $\sigma : \bar{\mathbb{F}}_p^{\times} \to \mu^{(p)}$, by the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p : \mathbb{F}_p)$ acting by composition on the right. One can describe Σ_p concretely by looking at the corresponding addition on $\tilde{\mu}^{(p)} = \{0\} \cup \mu^{(p)}$ which is inherited from the given isomorphism with $\bar{\mathbb{F}}_p$. It suffices to specify the addition with 1 and this shows that Σ_p is the set of bijections s of $\tilde{\mu}^{(p)} = \{0\} \cup \mu^{(p)}$ which commute with all their conjugates under rotations by elements of $\mu^{(p)}$, and fulfill $s(0) = 1$, $s \circ s \circ \ldots \circ s = id$ with p factors.

Mapping torus Y_p

But the comparison with the adèle class space shows that the space Σ_p is not what one wants as a fiber over p and one expects a finer space, which is the mapping torus Y_p of the action of the Frobenius on X_p. More concretely Y_p is the quotient

$$Y_p = (X_p \times (0,1))\,/\theta^{\mathbb{Z}}$$

of the product $X_p \times (0,1)$ of X_p by the open interval $(0,1) \subset \mathbb{R}$ by θ where

$$\theta(\sigma,\rho) = (\sigma \circ \mathrm{Fr}, \rho^p)\,, \quad \forall \sigma \in X_p,\ \rho \in (0,1).$$

Gluing together the Y_p

A. Weil showed how to construct the Weil group which compensates, at the Galois level, the absence of the connected component of identity in the idèle class group. We face here a similar problem at the level of the "curve". One should then perform the gluing of the fibers Y_p for different primes since as explained earlier, one needs to embed all these fibers in the same non-commutative space to account for the transversality factors in the explicit formulas.

The endomotive $\mathbb{F}_{1^\infty} \otimes_{\mathbb{F}_1} \mathbb{Q}$

Following a proposal of Soulé for $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z}$, we noted that

$$\mathbb{F}_{1^\infty} \otimes_{\mathbb{F}_1} \mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$$

is the abelian part of the BC-system. What matters is that, with this description of the BC-system as an *endomotive* $\mathcal{E}$, one can consider its points over any ring A and for a $\mathbb{Q}$-algebra this is just

$$\mathcal{E}(A) = \text{Hom}(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}], A)\,.$$

One can describe the space X_p as

$$X_p = \text{Hom}(\mathbb{Q}^{\text{cycl,p}}, \overline{\mathbb{Q}_p^{\text{un}}}) = \text{Hom}(\mathbb{Q}^{\text{cycl,p}}, \mathbb{C}_p)\,.$$

One obtains in this way, for each p the inclusions

$$X_p \subset \text{Hom}(\mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \overline{\mathbb{Q}_p^{\text{un}}}) = \mathcal{E}(\overline{\mathbb{Q}_p^{\text{un}}}) = \mathcal{E}(\mathbb{C}_p)\,.$$

By comparison with the adèle class space $\mathbb{H}_{\mathbb{Q}}$, one finds that the natural noncommutative space in which the above fiber Y_p embeds naturally is the quotient

$$(\mathcal{E}(\mathbb{C}_p) \times (0,1])\,/\mathbb{N}$$

where the action of n (in the semi-group $\mathbb{N}$) is the product of the action of the n-th Frobenius of the endomotive by the map $x \mapsto x^n$ in the interval $(0,1] \subset \mathbb{R}$.

Characteristic 1

A commutative semi-ring is called multiplicatively cancellative when the multiplication by any non-zero element is injective. Characteristic 1 means

$$x + x = x\,, \quad \forall x$$

Let R be a multiplicatively-cancellative commutative semi-ring of characteristic 1, then for any integer $n \in \mathbb{N}$, the map $x \mapsto x^n$ is an injective endomorphism of R.

A multiplicatively-cancellative commutative semi-ring of characteristic 1 is called *perfect* if the endomorphism $R \to R$, $x \mapsto x^n$ is surjective for all n.

Fractional powers

$$\vartheta_\alpha : R \to R, \qquad \vartheta_\alpha(x) = x^\alpha, \quad \forall \alpha \in \mathbb{Q}_+^*.$$

Then, by construction, the ϑ_α's are automorphisms $\vartheta_\alpha \in$ Aut(R) for $\alpha \in \mathbb{Q}_+^*$ and they fulfill the following properties

- $\vartheta_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in R$.

- $\vartheta_\lambda \circ \vartheta_\mu = \vartheta_{\lambda\mu}$ for all $\lambda, \mu \in \mathbb{Q}_+^*$.

- $\vartheta_\lambda(x)\vartheta_\mu(x) = \vartheta_{\lambda+\mu}(x)$ for all $\lambda, \mu \in \mathbb{Q}_+^*$ and $x \in R$.

Witt in characteristic p

R a strict p-ring with $S = R/pR$ and let τ be the Teichmüller lift $\tau : S \to R$. We consider the ring $S[[T]]$ of formal series in T with coefficients in S.

$$\mathbb{F}_p[[T]] \subset S[[T]]$$

We use τ to get a bijection of $S[[T]]$ with R,

$$\tilde{\tau}(\sum s_n T^n) = \sum \tau(s_n) p^n \in R$$

where the sum is p-adically convergent.

Theorem :

For any $x, y \in S$ one has

$$\tau(x) + \tau(y) = \tilde{\tau}(\sum_{\alpha \in I_p} w_p(\alpha) x^\alpha y^{1-\alpha})$$

Coefficients $w(\alpha)$ in characteristic one

To obtain the analogue of the Witt construction in characteristic 1, one looks for $w(\alpha)$ for $\alpha \in I = \mathbb{Q} \cap [0,1]$ giving associative and commutative operation

$$x +' y = \sum_{\alpha \in I} w(\alpha) x^{\alpha} y^{1-\alpha}$$

Besides

$$w(1-\alpha) = w(\alpha)$$

one needs

$$w(\alpha) w(\beta)^{\alpha} = w(\alpha\beta) w(\gamma)^{(1-\alpha\beta)}, \quad \gamma = \frac{\alpha(1-\beta)}{1-\alpha\beta}.$$

Positive solution

Let G be a uniquely divisible ordered abelian group such that $x \mapsto x^\alpha$ extends to an action of $\mathbb{R}^\times$. Let $w : I \to G$ be a solution such that

$$w(\alpha) \geq 1\,, \quad \forall \alpha \in I$$

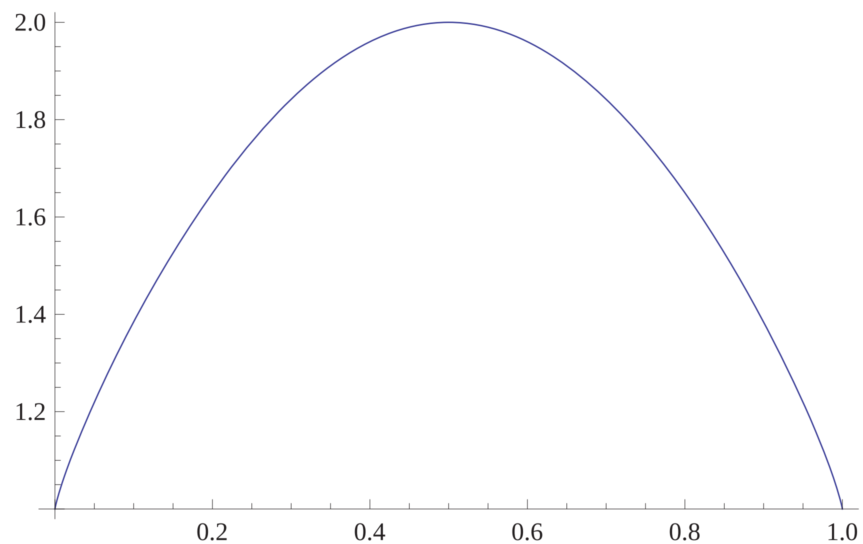
Then there exists $\rho \in G, \rho \geq 1$ such that

$$w(\alpha) = \rho^{-\alpha \log \alpha - (1-\alpha)\log(1-\alpha)}\,, \quad \forall \alpha \in I$$

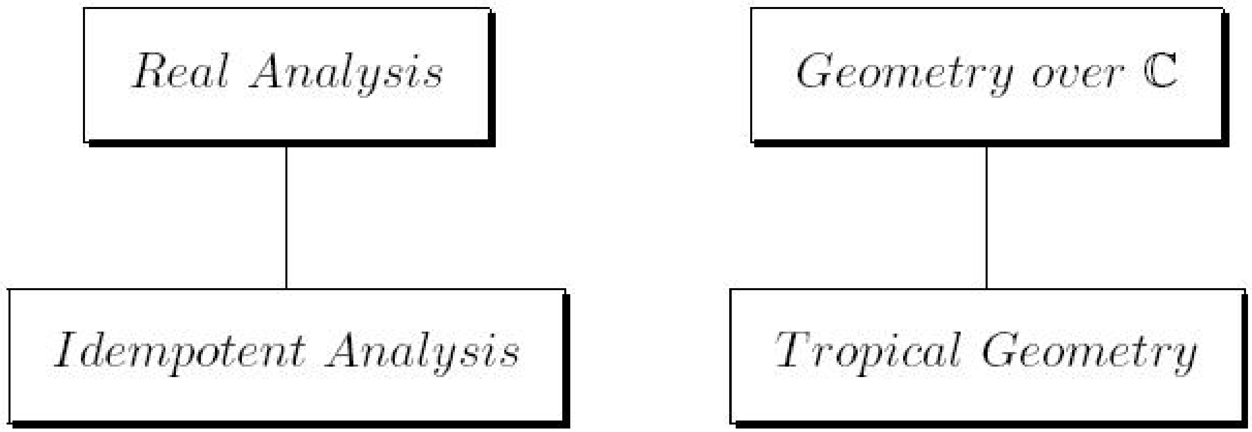
Entropy

$$S(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$$

$$\log(e^a + e^b) = \sup_{x\in[0,1]} S(x) + xa + (1-x)b$$



Graph of $\mathrm{Exp}(S(\alpha)) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$



One considers the semiring $\mathbb{R}_+ \supset \mathbb{B}$ with addition

$$(x, y) \mapsto \max(x, y)$$

and the one-parameter group $\theta_\lambda \in \mathrm{Aut}(\mathbb{R}_+^{max})$, $\theta_\lambda(x) = x^\lambda$.

Towards $\mathbb{R}^{un}$

To eliminate ρ it is natural to allow all values of ρ, *i.e.* to introduce a parameter $T \geq 0$,

$$\rho = e^T \in \mathbb{R}_+^{max}, \qquad \rho \geq 1$$

$w(\alpha)$ depends on T as it does in the Witt case :

$$w(\alpha, T) = \alpha^{-T\alpha}(1-\alpha)^{-T(1-\alpha)}$$

This means that one works with functions

$$f(T) \in \mathbb{R}_+^{max}$$

with the usual pointwise product and the new addition

$$(f_1 +_w f_2)(T) = \sum_{\alpha \in \bar{I}} w(\alpha, T) f_1(T)^\alpha f_2(T)^{1-\alpha}$$

Lemma

The addition $+_w$ is given by

$$(f_1 +_w f_2)(T) = (f_1(T)^{1/T} + f_2(T)^{1/T})^T$$

for $T > 0$ and by

$$(f_1 +_w f_2)(0) = \sup(f_1(0), f_2(0))$$

The sum of n terms x_j independent of T is given by

$$x_1 +_w \cdots +_w x_n = \left(\sum x_j^{1/T}\right)^T$$

$$1 +_w 1 +_w \cdots +_w 1 = n^T$$

Teichmüller lift

The constant functions $T \mapsto x$ are the analogue of the Teichmüller representatives

$$\tau(x)(T) = x \qquad \forall T$$

One has

$$\tau(x) + \tau(y) = \sum_{\alpha \in \bar{I}} w(\alpha, T)\, x^{\alpha} y^{1-\alpha}$$

where the sum in the right hand side is computed in $\mathbb{R}_{+}^{\max}$.

Residue morphism

The evaluation at $T = 0$ is by construction a morphism

$$\epsilon : f \mapsto f(0) \in \mathbb{R}_+^{\max}$$

We view this morphism as the analogue of the canonical map which exists for any strict p-ring

$$\epsilon_p : \mathbb{W}_p(K) \to K = \mathbb{W}_p(K)/p\mathbb{W}_p(K)$$

The automorphisms $\alpha_\lambda \in \mathrm{Aut}(\mathbb{R}^{un})$

One has a natural one parameter group of automorphisms α_λ which corresponds to the $\theta_\lambda \in \mathrm{Aut}(\mathbb{R}_+^{max})$. It is given by

$$\alpha_\lambda(f)(T) = f(T/\lambda)^\lambda \qquad \forall \lambda \in \mathbb{R}_+^\times$$

One has

$$\epsilon \circ \alpha_\lambda = \theta_\lambda \circ \epsilon\,, \quad \forall \lambda$$

and

$$\alpha_\lambda \circ \tau = \tau \circ \theta_\lambda\,, \quad \forall \lambda.$$

Moreover the fixed points of α_λ are of the form

$$f(T) = a^T$$

and they give the semi-field $\mathbb{R}_+$.

Characters and representation by functions

For each $T > 0$ the algebraic operations on the value $f(T)$ are the same as in the semi-field $\mathbb{R}_+$ using the evaluation $f(T)^{1/T}$. Thus there is a uniquely associated character χ_T which is such that

$$\chi_T(f) = f(T)^{1/T}$$

and we use the characters χ_T to represent the elements of the extension $\mathbb{R}^{\mathrm{un}}$ as functions of T with the ordinary operations of pointwise sum and product.

The extension $\mathbb{R}^{\text{un}}$

This shows that after symmetrization and passing to the field of quotients $\mathbb{R}^{\text{un}}$ contains at least the fractions of the form (in the χ representation)

$$\chi(f)(T) = \left(\sum \lambda_j e^{-s_j/T}\right) / \left(\sum \mu_j e^{-t_j/T}\right)$$

Deformation parameter $T = \hbar$

The key example is given by the functional integral in the Euclidean formulation of Quantum Field Theory. Indeed the generating function of Euclidean Green's functions is given by

$$Z(J_E) = \mathcal{N} \int \exp\left(-\frac{S(\phi_E) - \langle J_E, \phi_E \rangle}{\hbar}\right) \mathcal{D}[\phi_E]$$

where $S(\phi_E)$ is the Euclidean action, in terms of the Euclidean classical fields ϕ_E, the *source* J_E is an element of the linear space dual to that of Euclidean classical fields and the normalization factor $\mathcal{N}$ is the inverse of

$$\int \exp\left(-\frac{S(\phi_E)}{\hbar}\right) \mathcal{D}[\phi_E]$$

Such integrals are typical sums involving $+_w$ where w is the function of $\hbar$ given by

$$w(\alpha, \hbar) = \alpha^{-\hbar\alpha}(1-\alpha)^{-\hbar(1-\alpha)}$$

but since the sums are infinite one needs to extend the entropy from finite partitions of 1 to infinite partitions.