Foliated spaces

A smooth subbundle F of TV is called *integrable* iff one of the following equivalent conditions is satisfied :

a) Every $x \in V$ is contained in a submanifold W of V such that

$$T_y(W) = F_y \qquad \forall\, y \in W\,.$$

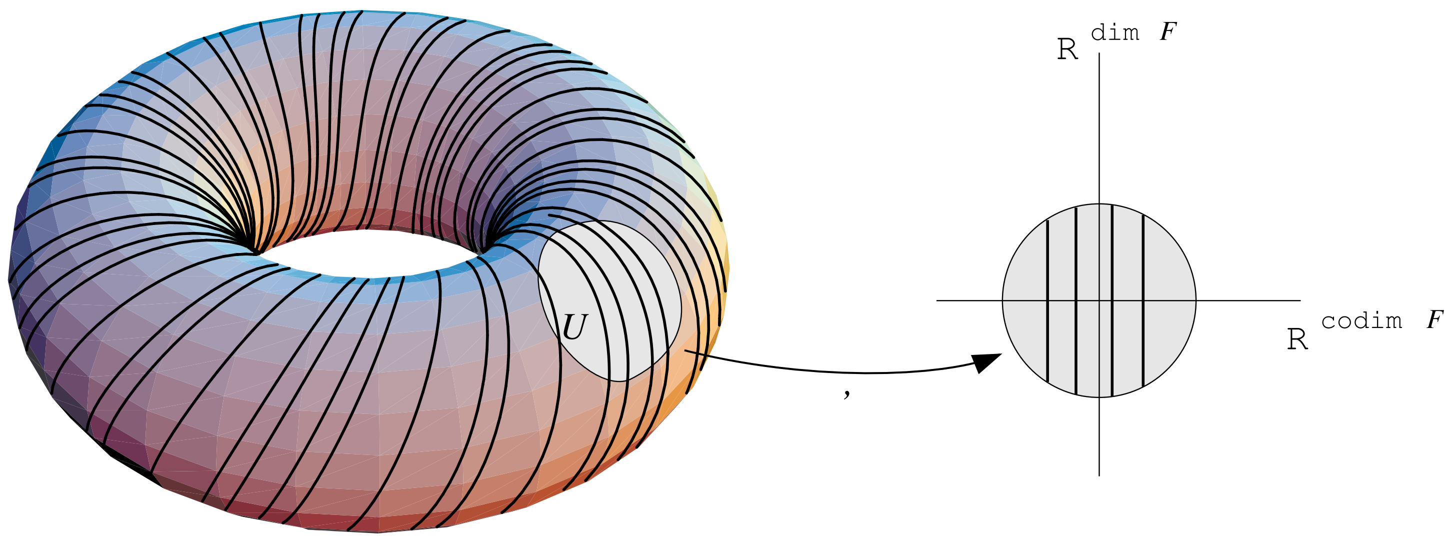
b) Every $x \in V$ is in the domain $U \subset V$ of a submersion $p : U \to \mathbb{R}^q$ ($q = \text{Codim}\, F$) with

$$F_y = \text{Ker}(p_*)_y \qquad \forall\, y \in U\,.$$

c) $C^\infty(F) = \{X \in C^\infty(TV)\,,\ X_x \in F_x \quad \forall\, x \in V\}$ is a Lie algebra.

d) The ideal $J(F)$ of smooth exterior differential forms which vanish on F is stable by exterior differentiation.

A foliation of V is given by an integrable subbundle F of TV. The leaves of the foliation (V,F) are the maximal connected submanifolds L of V with $T_x(L) = F_x$, $\forall\, x \in L$, and the partition of V in leaves $V = UL_\alpha$, $\alpha \in A$ is characterized geometrically by its "local triviality" : every point $x \in V$ has a neighborhood U and a system of local coordinates (such charts are called *foliation charts*) $(x^j)_{j=1,\ldots,\dim V}$ so that the partition of U in connected components of leaves (called *plaques*, they are the leaves of the restriction of F) corresponds to the partition of $\mathbb{R}^{\dim V} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\mathrm{Codim}\, F}$ in the parallel affine subspaces $\mathbb{R}^{\dim F} \times \mathrm{pt}$.



U
,
R dim F
R codim F

In the simplest examples, such as the Kronecker foliation of the 2-torus $V = \mathbb{R}^2/\mathbb{Z}^2$ given by the differential equation $dx = \theta\, dy$ where $\theta \notin \mathbb{Q}$, one sees that :

1) Though V is compact, the leaves L_α, $\alpha \in A$ can fail to be compact.

2) The space A of leaves L_α, $\alpha \in A$ can fail to be Hausdorff and in fact the quotient topology can be trivial (with no non trivial open subset).

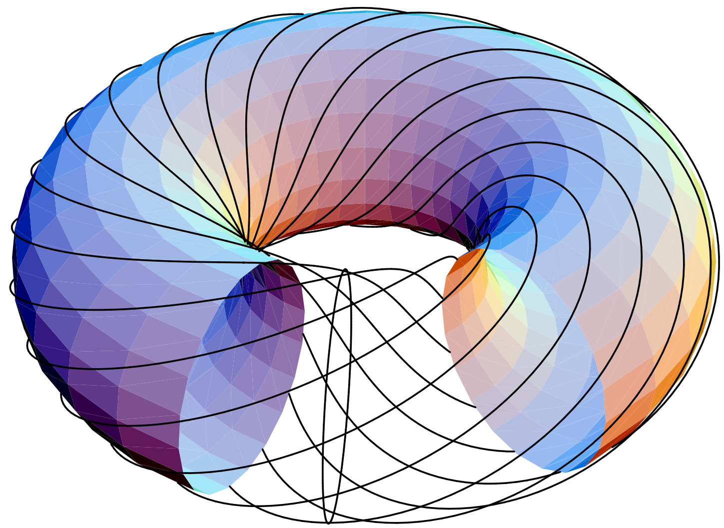
Noncommutative Quotient

When passing to the quotient, if we just consider invariant functions we obtain a very poor algebra of functions, since, even at the measurable level, we would only have constant functions. If instead we consider the noncommutative algebra of functions obtained by the general recipe of "noncommutative quotients" (functions on the graph of the equivalence relation with the convolution product), we obtain a very interesting and highly non–trivial algebra of functions describing the space of leaves of the foliation.

This is given (in the topological category) by the “irrational rotation algebra”, *i.e.* the C^*-algebra

$$\mathcal{A}_\theta := \{\Big(a_{ij}\Big) \quad i,j \in T \ \text{ in the same leaf } \}.$$

Namely, elements in the algebra $\mathcal{A}_\theta$ associated to the transversal $T \simeq S^1$ are just matrices (a_{ij}) where the indices are arbitrary pairs of elements i, j of S^1 belonging to the same leaf.



Noncommutative torus $C^\infty(\mathbb{T}^2_\theta)$

$\mathcal{A}_\theta$ is generated by two unitaries (U, V) with presentation given by the relation

$$VU = \lambda UV, \quad \text{with} \quad \lambda = \exp(2\pi i\theta)\,.$$

If we work in the smooth category, then a generic element b is given by a power series

$$b = \sum_{\mathbb{Z}^2} b_{nm} U^n V^m \ \in \mathcal{S}(\mathbb{Z}^2)$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$.

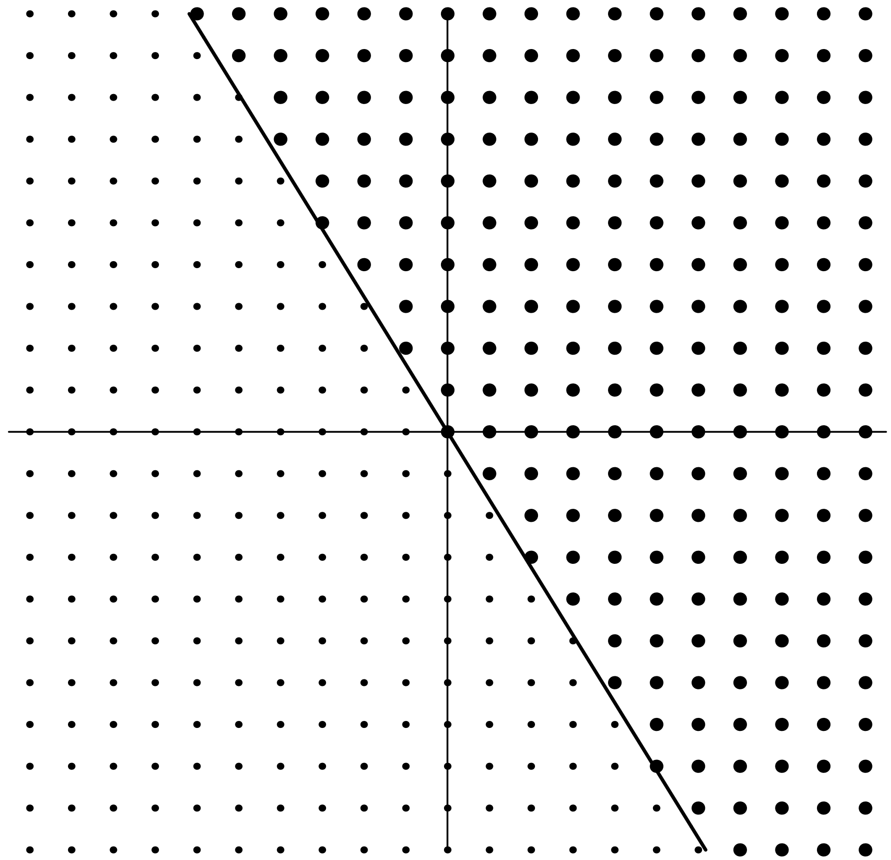
Smooth structure

The tangent space to the ordinary torus $\mathbb{T}^2$ is spanned by the tangent directions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. By choosing coordinates U, V, with $U = e^{2\pi i x}$ and $V = e^{2\pi i y}$, the tangent vectors are given by $\frac{\partial}{\partial x} = 2\pi i U \frac{\partial}{\partial U}$ and $\frac{\partial}{\partial y} = 2\pi i V \frac{\partial}{\partial V}$. These have analogs in terms of derivations of the algebra of the noncommutative torus. The two commuting vector fields which span the tangent space for an ordinary (commutative) 2-torus correspond algebraically to two commuting derivations of the algebra of smooth functions.

These derivations continue to make sense when we replace the generators U and V of $C^\infty(\mathbb{T}^2)$ by the generators of the algebra $C^\infty(\mathbb{T}^2_\theta)$, which no longer commute. The derivations are still given by the same formulas as in the commutative case,

$$\delta_1 = 2\pi i U \frac{\partial}{\partial U} \qquad \delta_2 = 2\pi i V \frac{\partial}{\partial V}$$

so that $\delta_1 \left(\sum b_{nm} U^n V^m\right) = 2\pi i \sum n b_{nm} U^n V^m$, and similarly for δ_2.



Finite projective modules over $\mathcal{A}_\theta$ are classified up to isomorphism by a pair of integers (p, q) such that $p + q\theta \geq 0$. For a choice of such pair, the corresponding module $\mathcal{H}^\theta_{p,q}$ is obtained from the transversal $T_{p,q}$ given by the closed geodesic of the torus $\mathbb{T}^2$ specified by (p, q), via the following construction. Elements of the module associated to the transversal $T_{p,q}$ are rectangular matrices, $(\xi_{i,j})$ with $(i, j) \in T \times S^1$, and with i and j belonging to the same leaf. The right action of $(a_{i,j}) \in \mathcal{A}_\theta$ is by matrix multiplication.

For instance, from the transversal $x = 0$ one obtains the following right module over $\mathcal{A}_\theta$. The underlying linear space is the usual Schwartz space

$$\mathcal{S}(\mathbb{R}) = \{\xi : \xi(s) \in \mathbb{C}, \quad \forall s \in \mathbb{R}\}$$

of complex-valued smooth functions on $\mathbb{R}$, all of whose derivatives are of rapid decay. The right module structure is given by the action of the generators U, V :

$$(\xi U)(s) = \xi(s + \theta) \qquad (\xi V)(s) = e^{2\pi i s}\xi(s) \quad \forall s \in \mathbb{R}\,.$$

One finds that the Schwartz space $\mathcal{S}(\mathbb{R})$ has dimension the real number

$$\dim_{\mathcal{A}}(\mathcal{S}) = \theta\,.$$

One similarly finds values $p + q\theta$ for the more general case. The appearance of a real-valued dimension is related to the *density* of transversals in the leaves, that is, the limit of

$$\frac{\# B_R \cap S}{\text{size of } B_R},$$

for a ball B_R of radius R in the leaf. In this sense, the dimension θ of the Schwartz space measures the relative densities of the two transversals $S = \{x = 0\}$ and $T = \{y = 0\}$.

Continuous dimension

In general, the appearance of non-integral dimension is a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when one looks from the measure theoretic point of view. The von Neumann algebra which describes the quotient space from the measure theoretic point of view is the well known hyperfinite factor of type II_1. In particular the classification of finite projective modules $\mathcal{E}$ over R is given by a positive real number, the Murray–von Neumann dimension

$$\mathrm{dim}_R(\mathcal{E}) \in \mathbb{R}_+ \,.$$

Connections and curvature

The analogs of connection and curvature of vector bundles are straightforward to obtain : a connection is just given by the associated covariant differentiation $\bigtriangledown$ on the space of smooth sections. Thus here it is given by a pair of linear operators on the Schwartz space of rapidly decaying functions,

$$\bigtriangledown_j : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

such that

$$\bigtriangledown_j(\xi b) = (\bigtriangledown_j \xi) b + \xi \delta_j(b) \quad \forall \xi \in \mathcal{S} \ , b \in \mathcal{A}_\theta \, .$$

Integrality

One checks that, as in the usual case, the trace of the curvature

$$\Omega = \nabla_1 \nabla_2 - \nabla_2 \nabla_1,$$

is independent of the choice of the connection.

We can make the following choice for the connection :

$$(\nabla_1 \xi)(s) = -\frac{2\pi i s}{\theta}\,\xi(s) \qquad (\nabla_2 \xi)(s) = \xi'(s)\,.$$

Notice that, up to the correct powers of $2\pi i$, the total curvature of $\mathcal{S}$ is an *integer*. In fact, the curvature Ω is constant, equal to $\frac{1}{\theta}$, so that the irrational number θ disappears in the total curvature, $\theta \times \frac{1}{\theta}$.

This integrality phenomenon, where the pairing of dimension and curvature (both of which are non–integral) yields an integer :

$$\dim \times \Omega \sim \theta \times \frac{1}{\theta} = \text{integer},$$

is the basis for the development of a theory of characteristic classes for noncommutative spaces. In the general case, this requires the development of more sophisticated tools, since analogs of the derivations δ_i used in the case of the noncommutative tori are not there in general. The general theory is obtained through cyclic homology.

Noncommutative Geometry

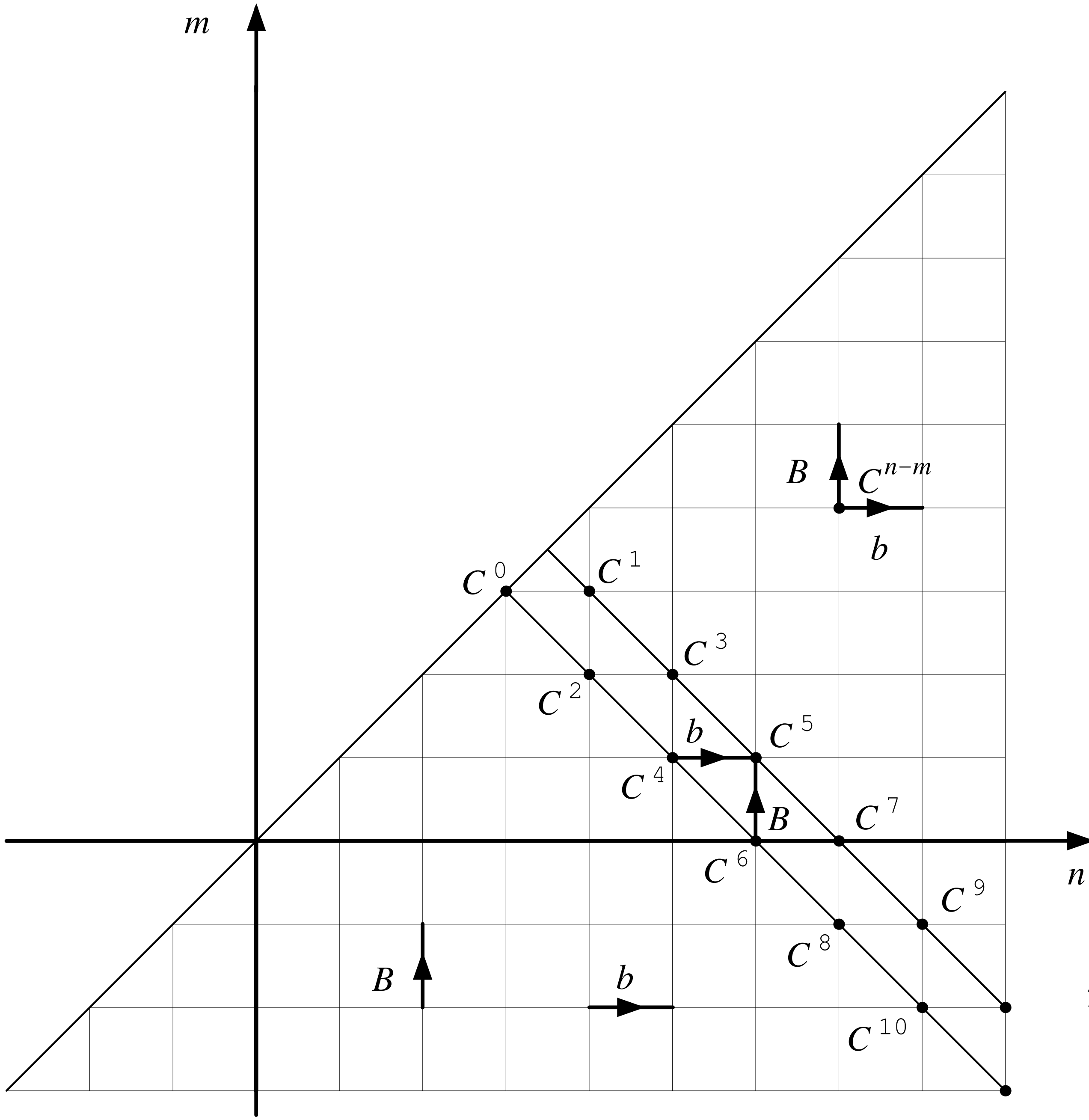
– Measure theory= von Neumann algebras (factors types) I, II, III_λ and flow of weights $W(M)$.
– K-theory.
– De Rham cohomology= cyclic cohomology $HC^n(A)$.
– Riemannian Geometry= Spectral triples

$$(A, \mathcal{H}, D)$$

Noncommutative Spaces

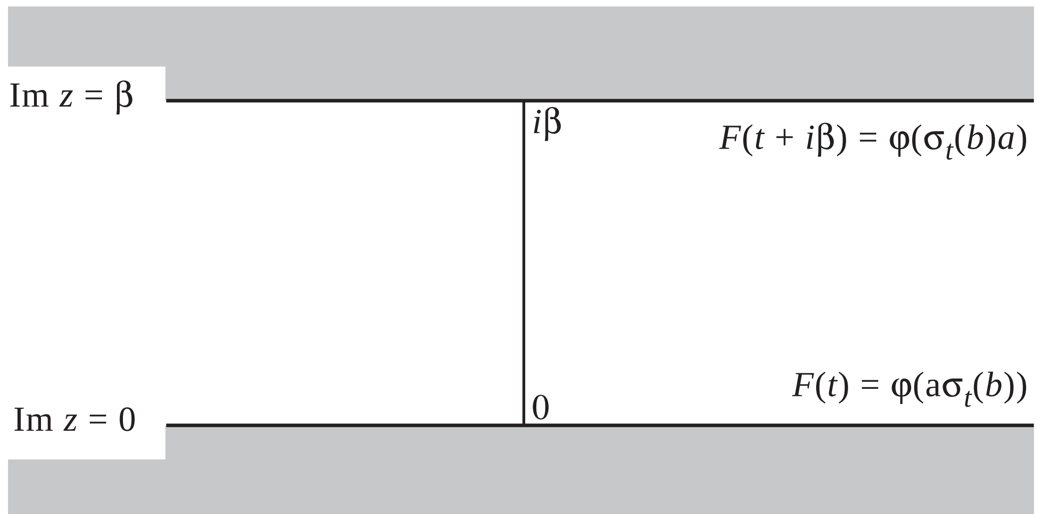
– Phase space in quantum mechanics (Heisenberg).
– Space of leaves of foliations (V, F).
– Irreducible representations of discrete group Γ.
– Quantum groups.

Two key examples are related to the distribution of prime numbers : the BC-system, and to the nature of space-time.



m
n
B
C^{n-m}
b
C^0
C^1
C^2
C^3
C^4
C^5
C^6
C^7
C^8
C^9
C^{10}

KMS Condition



$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \quad F_{x,y}(t+i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Tomita–Takesaki

Theorem

Let M be a von Neumann algebra and φ a faithful normal state on M, then there exists a unique

$$\sigma_t^\varphi \in \mathrm{Aut}(M)$$

which fulfills the KMS condition for $\beta = 1$.

Thesis (1971–1972)

Theorem (ac)

$$1 \to \mathrm{Int}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}) \to \mathrm{Out}(\mathcal{M}) \to 1,$$

The class of σ_t^{φ} in Out($\mathcal{M}$) does not depend on φ.

Thus a von Neumann algebra $\mathcal{M}$, has a canonical evolution

$$\mathbb{R} \xrightarrow{\delta} \mathrm{Out}(\mathcal{M}).$$

Noncommutativity $\Rightarrow$ Evolution

Godbillon-Vey invariant

$$d\omega = \beta \wedge \omega\,, \quad \alpha = d\beta \wedge \beta\,.$$

Theorem (ac) Let (V, F) be a 3 dimensional oriented, transversally oriented compact foliated manifold, ($\dim F = 2$). Let M be the associated von Neumann algebra, and $W(M)$ be its flow of weights. Then if the Godbillon-Vey invariant of (V, F) is different from 0, there exists an invariant probability measure for the flow $W(M)$.

General principles

Up to Morita equivalence the algebra of the foliation only depends upon the leaf space. The classifying situation is when the leaves are contractible.

- Cohomology along the leaves and longitudinal index theorem.
- Transversals and geometric group for K-theory of foliation.
- Cyclic cohomology of the foliation algebra and bicomplex with longitudinal and transverse differentials.

Foliations and the projective Adèle class space

A. Connes, C. Consani

Interpretation by Guillemin of the Selberg trace formula. V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J., Vol. 44, 3 (1977), 485-517.

It is this paper which motivated the adèle class space in 1996.

Anosov foliation

One takes Γ cocompact :

$$M = \mathrm{SL}_2(\mathbb{R})/\Gamma$$

Vector fields $\eta^{\pm}$ and ξ on M associated to

$$E^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The contact structure given by the 1-form α dual to ξ

$$\langle \xi, \alpha \rangle = 1\,, \quad \langle \eta^{\pm}, \alpha \rangle = 0$$

is preserved by ξ and $\alpha \wedge d\alpha \neq 0$.

Polarisation

One takes the Legendre foliation of the action of η^+. It is a globally invariant polarization for ξ.

$$C^\infty(M, \mathcal{M}^{\frac{1}{2}}) \xrightarrow{d} C^\infty(M, \mathcal{M}^{\frac{1}{2}} \otimes \wedge^1)$$

where d is leafwise differentiation and $\mathcal{M}^{\frac{1}{2}}$ the transverse half-densities.

M is the cosphere bundle S^*X where X is the Riemann surface $\mathbb{H}/\Gamma$ and ξ generates the geodesic flow.

$\mathbb{P}^1$ and PGL_2/B

$$\mathbb{P}^1_{\mathbb{F}_1}(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*) \subset \mathbb{K}^*\backslash \mathrm{PGL}_2(\mathbb{A}_{\mathbb{K}})/B(\mathbb{A}_{\mathbb{K}})$$

Maximal torus :

$$\mathbb{K}^* = \mathcal{T}(\mathbb{K}) \subset \mathcal{T}(\mathbb{A}_{\mathbb{K}}), \quad \mathcal{T}(\mathbb{A}_{\mathbb{K}})/\mathcal{T}(\mathbb{K}) \sim C_{\mathbb{K}}$$

Normalizer :

$$\mathcal{N}(\mathbb{A}_{\mathbb{K}}) \supset \mathcal{T}(\mathbb{K}), \quad \mathcal{N}(\mathbb{A}_{\mathbb{K}})/\mathcal{T}(\mathbb{K}) \sim C_{\mathbb{K}} \ltimes W$$

The quotient of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{K}})/B(\mathbb{A}_{\mathbb{K}})$ by $\mathbb{K}^*$ is the leaf space of the foliation of $\mathcal{T}(\mathbb{K})\backslash \mathrm{PGL}_2(\mathbb{A}_{\mathbb{K}})$ by the action of $B(\mathbb{A}_{\mathbb{K}})$.

Čech and sheaf cohomology for $\mathfrak{Mo}$-schemes

Lemma : Let X be an $\mathfrak{Mo}$-scheme.

1) Let $U \subset X$ be an affine open set, then

$$H^p(U, \mathcal{F}) = 0 \,, \quad \forall p > 0 \,, \quad \mathcal{F} \in \text{obj } \mathbb{A}(X)$$

2) Let $\mathcal{U} = (U_j)$ be an open cover of X such that all finite intersections $\bigcap U_k$ are affine, then for any sheaf $\mathcal{F}$ on X, one has

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathcal{U}, \mathcal{F}) \,, \quad \forall p \geq 0$$

where the right hand side is the Čech cohomology relative to the covering $\mathcal{U}$.

Projection $\underline{X}(M) \to X$

Lemma : Let X be an $\mathfrak{Mo}$-scheme.

1) For any monoid M there exists a canonical map

$$\pi_M : \underline{X}(M) \to X$$

such that

$$\pi_M(\phi) = \phi(\mathfrak{p}_M), \quad \forall \phi \in \mathrm{Hom}_{\mathfrak{Mo}}(\mathrm{Spec}(M), X)$$

2) Let U be an open subset of X and $\underline{U}$ the associated subfunctor of $\underline{X}$, then

$$\underline{U}(M) = \pi_M^{-1}(U) \subset \underline{X}(M)$$

The monoid $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

The idèle class group $C_{\mathbb{K}}$ of the Weil formula is the group M^* of invertible elements of the monoid

$$M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*\,,\ \mathbb{K}^* = \mathsf{GL}_1(\mathbb{K})$$

of adèle classes.

Projective adèle class space

One takes the $\mathfrak{Mo}$-scheme $\mathbb{P}^1_{\mathbb{F}_1}$ and a global field $\mathbb{K}$ and $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$. The projection :

$$\pi_M \,:\, \mathbb{P}^1_{\mathbb{F}_1}(M) = M \cup_{M^*} M \to \mathbb{P}^1_{\mathbb{F}_1}$$

one has

$$\mathbb{P}^1_{\mathbb{F}_1} = \{0, u, \infty\}\,, \quad \overline{\{0\}} = \{0\}\,,$$

$$\overline{\{u\}} = \mathbb{P}^1_{\mathbb{F}_1}\,, \quad \overline{\{\infty\}} = \{\infty\}$$

and three open sets $U^{\pm}$ with $U = U^+ \cap U^-$ in $\mathbb{P}^1_{\mathbb{F}_1}$

$$U_+ = \mathbb{P}^1_{\mathbb{F}_1} \backslash \{\infty\}\,, \quad U_- = \mathbb{P}^1_{\mathbb{F}_1} \backslash \{0\}\,, \quad U = U_+ \cap U_-$$

Functions on $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

To obtain a natural space $\mathcal{S}(M)$ of functions on M on lets $\mathbb{K}^*$ act by $f_q(x) = f(qx)$ on the Bruhat-Schwartz space $\mathcal{S}(\mathbb{A}_{\mathbb{K}})$.

$$0 \to \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \to \mathcal{S}(\mathbb{A}_{\mathbb{K}}) \stackrel{\epsilon}{\to} \mathbb{C} \oplus \mathbb{C}[1] \to 0$$

where $\epsilon(f) = (f(0), \int f(x)dx) \in \mathbb{C} \oplus \mathbb{C}[1]$ is $\mathbb{K}^*$-invariant and one lets

$$\mathcal{S}(M) = \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0/\{f - f_q\} \oplus \mathbb{C} \oplus \mathbb{C}[1]$$

Sheaf Ω over $\mathbb{P}^1_{\mathbb{F}_1}$

$$\begin{aligned} \Gamma(U_+,\Omega) &= \mathcal{S}(M) \\ \Gamma(U_-,\Omega) &= \mathcal{S}(M) \\ \Gamma(U_+\cap U_-,\Omega) &= \mathbf{S}\,(C_{\mathbb{K}}) \end{aligned}$$

and restriction maps :

$$\begin{aligned} (\mathrm{Res}\, f)(g) &= \sum_{q\in\mathbb{K}^*} f(qg)\,, \quad \forall f\in\Gamma(U_+,\Omega) \\ (\mathrm{Res}\, h)(g) &= |g|^{-1}\sum_{q\in\mathbb{K}^*} h(qg^{-1})\,, \quad \forall h\in\Gamma(U_-,\Omega) \end{aligned}$$

Čech Complex

The Čech Complex of the cover $\mathcal{U} = \{U_\pm\}$ is

$$\begin{aligned} C^0 &= \Gamma(U_+, \Omega) \times \Gamma(U_-, \Omega) \\ C^1 &= \Gamma(U_+ \cap U_-, \Omega) \end{aligned}$$

The coboundary $\partial : C^0 \to C^1$ is

$$\partial(f, h)(g) = \Sigma(f)(g) - |g|^{-1}\Sigma(h)(g^{-1}) \in C^1.$$

where

$$\Sigma(f)(g) = \sum_{q \in \mathbb{K}^*} f(qg)$$

Poisson Formula

Let $\mathbb{K}$ be a global field, α a nontrivial character of $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$. The lattice $\mathbb{K} \subset \mathbb{A}_{\mathbb{K}}$ is its own dual.

$$F(f)(a) = \int f(x)\alpha(ax)dx$$

$$\sum f(q) = \sum Ff(q)$$

With

$$f(x) = h(g^{-1}x)\,,\;\; Ff(a) = |g|Fh(ga)$$

one gets

$$\sum h(g^{-1}q) = |g| \sum F(h)(gq)$$

Note that F is canonical.

Fourier–Poisson

Lemma : The kernel of $\partial : C^0 \to C^1$ is the graph of Fourier transform

$$H^0(\mathbb{P}^1_{\mathbb{F}_1}, \Omega) = \{(f, F(f)) \mid f \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0/\{f - f_q\}\}$$

$$\oplus 2\tau \oplus 2\tau[1]$$

Let $(f, h) \in \operatorname{Ker}\partial$. Poisson gives

$$\Sigma(Fh)(g) = |g|^{-1}\Sigma(h)(g^{-1})$$

and thus

$$\Sigma(f - Fh) = 0$$

so that $f - Fh$ is in the coinvariants.

Representation of $C_{\mathbb{K}}$

$$\begin{aligned}
\vartheta_+(\lambda)f(x) &= f(\lambda^{-1}x)\,, \quad \forall f \in \Gamma(U_+, \Omega) \\
\vartheta_-(\lambda)f(x) &= |\lambda| f(\lambda x)\,, \quad \forall f \in \Gamma(U_-, \Omega) \\
\vartheta(\lambda)f(x) &= f(\lambda^{-1}x)\,, \quad \forall f \in \Gamma(U_+ \cap U_-, \Omega)
\end{aligned}$$

defines an action of $C_{\mathbb{K}}$ on the sheaf Ω

$$\mathrm{Res}(\vartheta_-(\lambda)f)(g) = |g|^{-1} \sum_{q \in \mathbb{K}^*} (\vartheta_-(\lambda)f)(qg^{-1})$$

$$= |g|^{-1} \sum_{q \in \mathbb{K}^*} |\lambda| f(\lambda q g^{-1})$$

and

$$\vartheta(\lambda)\mathrm{Res}(f)(g) = \mathrm{Res}(f)(\lambda^{-1}g) = |\lambda^{-1}g|^{-1} \sum_{q \in \mathbb{K}^*} f(q(\lambda^{-1}g)^{-1})$$

Action of W

The Weyl group W acts on $\mathbb{P}^1_{\mathbb{F}_1}$,

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

exchanging 0 and ∞.

The action of w on Ω is :

$$\begin{aligned} w^{\#} f &= f \in \Gamma(U_-, \Omega)\,, \quad \forall f \in \Gamma(U_+, \Omega) \\ w^{\#} f &= f \in \Gamma(U_+, \Omega)\,, \quad \forall f \in \Gamma(U_-, \Omega) \\ w^{\#} f(g) &= |g|^{-1} f(g^{-1})\,, \quad \forall f \in \Gamma(U_+ \cap U_-, \Omega) \end{aligned}$$

Action of $N = C_{\mathbb{K}} \ltimes W$

Proposition : There is a unique action of $N = C_{\mathbb{K}} \ltimes W$ on Ω given as above on W and on $C_{\mathbb{K}}$ by the twist $\vartheta[-\frac{1}{2}]$.

$$(\vartheta(\lambda) w^{\#} f)(g) = |\lambda| (w^{\#} \vartheta(\lambda^{-1}) f)(g)$$

Spectral realization and $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$

Theorem (CC) : The cohomology $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ gives the spectral realization of zeros of L-functions. The spectrum of the action of $C_{\mathbb{K}}$ on $H^1(\mathbb{P}^1_{\mathbb{F}_1}, \Omega)$ by $\vartheta[-\frac{1}{2}]$ is invariant by the symmetry $\chi(g) \mapsto \chi(g^{-1})$ of Grössencharakters of $\mathbb{K}$.

The sheaf $\Omega[-\frac{1}{2}]$ is that of half-densities :

$$f(x)|dx|^{\frac{1}{2}} \sim f(x)|x|^{\frac{1}{2}}|d^*x|^{\frac{1}{2}}$$

Fixed points of $C_{\mathbb{K}}$ on $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

Let $(x, u) \in M \times C_{\mathbb{K}} \,|\, u\, x = x$, with $u \neq 1$. There exists $v \in \Sigma_{\mathbb{K}}$ with

$$x \in \mathfrak{p}_v = \{x \in M \,|\, x_v = 0\}$$

The isotropy group of any $x \in \mathfrak{p}_v$ contains

$$\mathbb{K}_v^* \subset C_{\mathbb{K}}\,, \quad \mathbb{K}_v^* = \{(k_w) \,|\, k_w = 1 \ \forall w \neq v\}$$

Trace of the action of $\mathbb{K}_v^*$ on $\mathbb{K}_v$

$$(T\,\xi)\,(x) = \int k(x,y)\,\xi(y)\,dy$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) = \int k(x,x)\,dx\,.$$

$$T\xi(x) = \xi(\lambda x)\,, k(x,y) = \delta(\lambda x - y)$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) = \int k(x,x)\,dx = \int \delta(\lambda x - x)\,dx$$

$$= \frac{1}{|\lambda - 1|}\int \delta(z)\,dz = \frac{1}{|\lambda - 1|}$$

(with $z = (\lambda - 1)x$)

Trace of the action of $C_{\mathbb{K}}$ on $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

The transverse space to $\mathfrak{p}_v \subset M$ is the local field $\mathbb{K}_v$. The isotropy group $I_v = \mathbb{K}_v^*$ acts as $\mathbb{K}_v^*$ on $\mathbb{K}_v$ by multiplication

$$\vartheta(u)\xi(x) = \xi(u^{-1}x)$$

$$\mathrm{Tr}_{\mathrm{distr}}\left(\int h(u)\vartheta(u)d^*u\right) = \sum_v \int_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u$$

Periodic orbits on $\mathbb{P}^1$

Passing to projective space gives a doubling of periodic orbits for the action of $C_{\mathbb{K}}$. One lets

– Λ regular representtaion of $C_{\mathbb{K}}$.

– τ et $\tau[1]$ trivial representation and its twist.

– $\tau[\chi, \rho]$ character associated to a zero $\rho \in Z_{\chi}$ of $L(\chi, s)$.

– $\int'_{\mathbb{K}_v^*}$ local contribution.

Weil :

$$\tau + \tau[1] - \sum \tau[\chi, \rho] = \sum \int'_{\mathbb{K}_v^*}$$

H^0 and H^1 for $\mathbb{P}^1$

$$H^0 = \Lambda + \sum \int'_{\mathbb{K}_v^*} + \tau + \tau[1]$$

$$H^1 = \sum \tau[\chi, \rho] = \tau + \tau[1] - \sum \int'_{\mathbb{K}_v^*}$$

$$H^0 - H^1 = \Lambda + 2 \sum \int'_{\mathbb{K}_v^*}$$

which accounts for the doubling of periodic orbits

$$\mathrm{Tr}_{C^0} = 2 \left(\Lambda + \sum \int'_{\mathbb{K}_v^*} \right) , \quad \mathrm{Tr}_{C^1} = \Lambda$$

$$\mathrm{Tr}_{C^0} - \mathrm{Tr}_{C^1} = \Lambda + 2 \sum \int'_{\mathbb{K}_v^*}$$