KOLYVAGIN SEMINAR: OVERVIEW AND BACKGROUND MATERIAL

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1. The Result

Let E be an elliptic curve defined over $\mathbb{Q}$, and suppose its L-function has a simple zero at s=1. Then the conjecture of Birch and Swinnerton-Dyer [B-S-D] predicts that E should have rank 1. Gross and Zagier [G-Z] were able to prove that E has rank at least 1, by proving the result first over imaginary quadratic fields K with the property that the conductor N of E splits in K, and then using existing theorems to apply their results to $\mathbb{Q}$. Explicitly, they did this by showing that a certain point y_K on E over K, constructed by Birch and dubbed a Heegner point [Bi], satisfies a formula giving its canonical height as a non-zero multiple of $L'(E/K, 1)$. Thus, when the zero is simple, the canonical height of y_K is non-zero, so E has rank at least one over K.

Using certain special properties of Heegner points, Kolyvagin was then able to apply his theory of Euler systems to extend these results rather dramatically. In particular, he proved that when $L(E/K, s)$ has a simple zero at $s = 1$, that in fact E has rank exactly 1, and furthermore, $Ш(E/K)$ is finite [Ko]. Benedict Gross' paper *Kolyvagin's work on modular elliptic curves* [Gr], which will be the focus of this seminar, gives an exposition of Kolyvagin's proof of a large part of this result, namely:

Theorem 1. *If E does not have complex multiplication, let p be an odd prime not dividing y_K on $E(K)$, and suppose further that the extension $\mathbb{Q}(E[p])$ is as large as possible (i.e., has Galois group $GL_2(\mathbb{Z}/p\mathbb{Z})$ over $\mathbb{Q}$). Then E has rank 1, and the p-part of $Ш(E/K)$ is trivial.*

Note that thanks to the exact sequence

$$0 \to E(K)/pE(K) \to \mathrm{Sel}_p(E/K) \to Ш(E/K)[p] \to 0$$

we can apply Gross-Zagier, which tells us that $E(K)/pE(K)$ is at least $\mathbb{Z}/p\mathbb{Z}$, to conclude that to prove the theorem it suffices to show that $\mathrm{Sel}_p(E/K) = \mathbb{Z}/p\mathbb{Z}$. Thus, this Selmer group calculation will be the focus of the seminar.

2. Heegner Point Background

Clearly, a good familiarity with elliptic curve theory and algebraic number theory will be necessary to even understand the statement of the results. Beyond this, there are specific topics that will be important, such as knowledge of some of the basic statements of class field theory, and the actions of Frobenius automorphisms and complex conjugation on torsion points of elliptic curves. These topics will be handled on a case by case basis, although it is safe to assume that at least some will be have to be presupposed.

However, to even start on Kolyvagin's Euler system construction, it is necessary to understand Heegner points. Defining Heegner points is quite simple, but showing that they are defined over the right field requires some work. The main prerequisite for this is the theory of complex multiplication, which develops the theory of lattices in $\mathbb{C}$ which are closed under multiplication by some complex number, and therefore said to have complex multiplication. This is then applied to relate values of certain modular functions at imaginary quadratic values to a collection of class fields of imaginary quadratic extensions of $\mathbb{Q}$ (called ring class fields). Given this theory, if E is an elliptic curve of conductor N, it has a Weil parametrization by $X_0(N)$, and the Heegner point y_K can be constructed rather easily: let x_1 be the point on $Y_0(N) = \mathcal{H}/\Gamma_0(N)$ determined by an appropriate generator of $\mathcal{O}_K \subset \mathbb{C}$, and y_1 the image of x_1 on E under the parametrization map, then y_K is the trace of the Galois conjugates of y_1. The theory of complex multiplication is needed for showing that x_1, given an appropriate algebraic model of $X_0(N)$, is defined over the Hilbert class field of K; it follows that y_1 is defined over the same field, giving a very explicit Galois group over which to take the trace.

To construct his Euler system, Kolyvagin tinkers with the notion of a Heegner point to take a system of points y_n, not just defined over the Hilbert class field of K, but also over a collection of ring class fields of K. In order to prove the Euler system axioms, which give relations between y_n and y_m where $n = lm$ for l prime, theory of a slightly different flavor is involved. Here, it is necessary to understand the basic techniques for manipulating modular elliptic curves. Thus, a knowledge of the theory of modular curves and modular forms, including the Hecke correspondence and Hecke operators, is prerequisite. The Eichler-Shimura congruence relation also plays a key role here.

3. The Euler System Argument

Having produced a system of points satisfying the axioms of an Euler system, Kolyvagin proceeds to apply the usual Euler system argument to bound the size of $\mathrm{Sel}_p(E/K)$. Basic Galois cohomology is prevalent throughout the argument. The first step is to use the y_n to produce various global Galois cohomology classes, including $d(n) \in H^1(K, E)[p]$, all with reasonably concrete conditions for them to be trivial.

Next, everything is refined by decomposing the relevant cohomology groups into eigenspaces for complex conjugation. Conditions are given for the $d(n)$ to be locally trivial. Tate Local Duality is used to set up a local nondegenerate pairing $\langle , \rangle$ between $E(K_\lambda)/pE(K_\lambda)$ and $H^1(K_\lambda, E)[p]$, which can also be decomposed by eigenspaces. Setting this pairing up involves some more technical cohomology results which may or may not require the introduction of cohomology theories beyond Galois cohomology to really understand. The pairing is then applied to show that if one can produce an appropriate cohomology class (suggestively called d) locally trivial everywhere except at λ, then any element of the Selmer group is locally trivial at λ. The proof of this also makes central use of the result from global class field theory that the sum of local invariants is zero.

Finally, some concrete work with the Selmer group relates both local and global triviality of its elements to statements about the vanishing of certain pairings $[,]$, so the basic strategy to complete the proof is to use the Chebotarev Density Theorem to produce primes l for which the cohomology classes $d(l)$ are locally trivial

everywhere except at one place, which by the earlier result forces the Selmer group to be locally trivial; then via the concrete work with the pairings $[,]$, this may be translated into the final desired global bounds on the Selmer group.

4. Summary of Background

These are the background topics required for Gross' paper, listed roughly in order of how likely they are to be left as prerequisites for the seminar:

- Some algebraic number theory
- Some elliptic curve theory
- Some Galois cohomology
- Basic statements of local and global class field theory
- Theory of complex multiplication
- Basic theory of modularity of elliptic curves
- More sophisticated cohomologies (may be avoidable)

Many of these topics function well as black boxes, so if we don't have time to cover all topics we don't already know, we should still have minimal trouble understanding the flow of the argument. For instance, we will certainly take the work of Gross and Zagier as a very important black box.

5. References

[Bi] B. Birch, Heegner Points of Elliptic Curves. In *Symposia Mathematica*, vol. 15 (1975), pp. 441-445.

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[Gr] B. Gross, Kolyvagin's Work on Modular Elliptic Curves. In *L-Functions and Arithmetic*, LMS Lecture Notes 153, London Mathematical Society, Cambridge, 1991, pp. 235-256.

[G-Z] B. Gross and D. Zagier, Heegner Points and Derivatives of L-Series. In *Inventiones Mathematicae*, vol. 44 (1986), pp. 225-320.

[Ko] V. Kolyvagin, Euler Systems. In *Grothendieck Festschrift vol. 2*, Birkhauser, Boston, 1990, pp. 435-483.