Let R be a ring and let $f_1, \ldots, f_n$ be elements that generate the unit ideal. Let M be an R-module. For $i_1, \ldots, i_k \in \{1, \ldots, n\}$, let $M_{i_1,\ldots,i_k}$ be the module M_f for $f = f_{i_1} \cdots f_{i_k}$. For $k = 0, 1, \ldots$, let C_k be the direct sum of $M_{i_1,\ldots,i_k}$ over all tuples $(i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_n \leq k$. Define the map $C_k \to C_{k+1}$ by

$$(s_{i_1,\ldots,i_k})_{i_1,\ldots,i_k=1}^n \to \left(\sum_{j=0}^{k} (-1)^j s_{i_1,\ldots,\widehat{i_j},\ldots,i_k} \right)_{i_1,\ldots,i_{k+1}=1}^n .$$

We prove that the sequence

$$0 \to C_0 \to C_1 \to \cdots$$

of R-modules is exact. (One gets a similar result if one allows $(i_1, \ldots, i_k)$ to run over all tuples in $\{1, \ldots, n\}^k$.)

It suffices to check that for each prime ideal $\mathfrak{p}$, the localization

$$0 \to C_{0,\mathfrak{p}} \to C_{1,\mathfrak{p}} \to \cdots$$

is exact. Since $f_1, \ldots, f_n$ generate the unit ideal, there exists an index i for which $\mathfrak{p} \in D(f_i)$ (i.e., $f_i \notin \mathfrak{p}$); without loss of generality (but with a bit of a headache in swapping signs around, which we skip over) we may assume $\mathfrak{p} \in f_1$.

We now study the effect of localizing at $\mathfrak{p}$. First of all, it doesn't matter in which order we invert things, so

$$(M_{i_1,\ldots,i_k})_{\mathfrak{p}} = (M_{\mathfrak{p}})_{f_{i_1} \cdots f_{i_k}}.$$

If $i_1 = 1$, then $f_1 \notin \mathfrak{p}$ and so localizing at $\mathfrak{p}$ already involves inverting f_1. In that case,

$$(M_{i_1,\ldots,i_k})_{\mathfrak{p}} = (M_{\mathfrak{p}})_{f_{i_1} \cdots f_{i_k}} = (M_{\mathfrak{p}})_{f_{i_2} \cdots f_{i_k}}.$$

Let C'_i be the analogue of C_i where the index 1 is disallowed. We can then rewrite our original sequence as

$$0 \to C'_0 \to C'_0 \oplus C'_1 \to C'_1 \oplus C'_2 \to \cdots$$

which is evidently exact.