Let $R$ be a ring and let $f_1, \ldots, f_n$ be elements that generate the unit ideal. Let $M$ be an $R$-module. For $i_1, \ldots, i_k \in \{1, \ldots, n\}$, let $M_{i_1, \ldots, i_k}$ be the module $M_f$ for $f = f_{i_1} \cdots f_{i_k}$. For $k = 0, 1, \ldots$, let $C_k$ be the direct sum of $M_{i_1, \ldots, i_k}$ over all tuples $(i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_n \leq k$. Define the map $C_k \to C_{k+1}$ by

$$(s_{i_1, \ldots, i_k})_{i_1, \ldots, i_k = 1}^n \to \left(\sum_{j=0}^{k} (-1)^j s_{i_1, \ldots, \hat{i}_j, \ldots, i_k}\right)_{i_1, \ldots, i_{k+1} = 1}^n.$$ 

We prove that the sequence

$$0 \to C_0 \to C_1 \to \cdots$$

of $R$-modules is exact. (One gets a similar result if one allows $(i_1, \ldots, i_k)$ to run over all tuples in $\{1, \ldots, n\}^k$.)

It suffices to check that for each prime ideal $p$, the localization

$$0 \to C_{0,p} \to C_{1,p} \to \cdots$$

is exact. Since $f_1, \ldots, f_n$ generate the unit ideal, there exists an index $i$ for which $p \in D(f_i)$ (i.e., $f_i \notin p$); without loss of generality (but with a bit of a headache in swapping signs around, which we skip over) we may assume $p \in f_1$.

We now study the effect of localizing at $p$. First of all, it doesn’t matter in which order we invert things, so

$$(M_{i_1, \ldots, i_k})_p = (M_p)_{f_1 \cdots f_{i_k}}.$$ 

If $i_1 = 1$, then $f_1 \notin p$ and so localizing at $p$ already involves inverting $f_1$. In that case,

$$(M_{i_1, \ldots, i_k})_p = (M_p)_{f_1 \cdots f_{i_k}} = (M_p)_{f_2 \cdots f_{i_k}}.$$ 

Let $C'_i$ be the analogue of $C_i$ where the index 1 is disallowed. We can then rewrite our original sequence as

$$0 \to C'_0 \to C'_0 \oplus C'_1 \to C'_1 \oplus C'_2 \to \cdots$$

which is evidently exact.