Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 1

1. Identify $\mathbb{R}^4$ with the space of quaternions $\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$. Then the function $(a+bi+cj+dk) \mapsto i(a+bi+cj+dk) = -b+ai-d+cj$ defines an everywhere nonzero section of the tangent bundle.

2. (a) Let $\{U_i\}_{i\in I}$ be an open cover of an open subset U of X. The map $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ is injective because a function $g : U \to Y$ is determined by its restrictions to any set-theoretic cover. To check the rest of the sheaf property, choose $g_i \in \mathcal{F}(U_i)$ such that $g_i|_{U_i\cap U_j} = g_j|_{U_i\cap U_j}$. Define the function $g : U \to Y$ by setting $g(x) = g_i(x)$ for any index $i \in I$ for which $x \in U_i$; this is well-defined because all choices of i give the same answer. Then $(f \circ g)(x) = (f \circ g_i)(x) = x$, so $f \circ g$ equals the inclusion $U \to X$. For any open subset W of Y, $g^{-1}(W) = \cup_{i\in I} g_i^{-1}(W)$ is open in U, so g is continuous. Hence $g \in \mathcal{F}(U)$, so $\mathcal{F}$ is a sheaf.

   (b) We define a basis of open sets as follows. For each open subset U of X and each $g \in \mathcal{G}(U)$, include the set $V_{g,U} = \{g_x : x \in U\}$. To check that this is a basis, we must check that $V_{g,U} \cap V_{g',U'}$ can be written as a union of open sets, or equivalently, for any point $y \in V_{g,U} \cap V_{g',U'}$ we can find a basic open subset of $V_{g,U} \cap V_{g',U'}$ containing y. Namely, put $x = f(y)$. The fact that $y \in V_{g,U} \cap V_{g',U'}$ means that $g_x = g'_x$, so on some open subset U'' of $U' \cap U''$ we have $g|_{U''} = g'|_{U''}$. Let $g'' \in \mathcal{G}(U'')$ be the restriction of g to U''; then $y \in V_{g'',U''} \subseteq V_{g,U} \cap V_{g',U'}$. This completes the proof that the $V_{g,U}$ form a basis.

   To check that $\mathcal{F} = \mathcal{G}$, we first check that for every open set $U \subseteq X$ and every $g \in \mathcal{G}(U)$, the corresponding function $\tilde{g} : U \to Y$ is continuous. It is enough to check that for every open set $V_{g',U'}$, the set $\tilde{g}^{-1}(V_{g',U'})$ is open. For each $x \in \tilde{g}^{-1}(V_{g',U'})$, we have $g_x = g'_x$, so there exists an open subset U'' of $U \cap U'$ such that $g|_{U''} = g'|_{U''}$. Then $x \in U'' \subseteq \tilde{g}^{-1}(V_{g',U'})$, so $\tilde{g}^{-1}(V_{g',U'})$ contains an open neighborhood of each of its points; therefore it is open.

   We next check that any continuous function $h : U \to Y$ corresponds to a section of $\mathcal{G}$. Since $\mathcal{G}$ is a sheaf, it is enough to check this locally around a point $x \in U$. By the definition of $\mathcal{G}_x$, we can find an open subset U' of U and a section $g \in \mathcal{F}(U')$ such that $g_x = h(x)$. Since h is continuous, $h^{-1}(V_{g',U'})$ is open; since it contains x, it also contains an open subset U'' of X containing x. But then the restriction of h to U'' corresponds to the section $g|_{U''} \in \mathcal{F}(U'')$, as claimed.

   (c) For U an open subset of X, we must check that $f^{-1}(U)$ is open. To see this, pick any $y \in f^{-1}(U)$ and put $x = f(y)$; then $y \in \mathcal{G}_x$. We can thus find an open subset U' of U and a section $g \in \mathcal{G}(U')$ such that $g_x = y$; then $V_{g,U} \subseteq f^{-1}(U)$. Therefore, $f^{-1}(U)$ contains an open neighborhood of each of its points, and so is open.

   Side remark: with this topology on Y, the map f is a *covering space* map in the sense of point-set topology.

3. Let $\mathbf{Pre}_X$ and $\mathbf{She}_X$ denote the categories of presheaves on X and sheaves on X, respectively. The adjoint property means the property: for $\mathcal{F} \in \mathbf{Pre}_X$ and $\mathcal{G} \in \mathbf{She}_X$, there is a bijection
$$\mathrm{Mor}_{\mathbf{Pre}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Mor}_{\mathbf{She}_X}(\tilde{\mathcal{F}}, \mathcal{G})$$
which is functorial in both $\mathcal{F}$ and $\mathcal{G}$.

   Let $f : \mathcal{F} \to \mathcal{G}$ be a map of presheaves. This then defines a map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks for each $x \in X$. To see that this defines a map of sheaves, we must check that for any open subset U of X and any $s \in \tilde{\mathcal{F}}(U)$, the elements $f_x(s_x) \in \mathcal{G}_x$ come from a section of $\mathcal{G}$. Since $\mathcal{G}$ is already a sheaf, it is enough to check this locally around an arbitrary point $x \in U$. But by the definition of $\tilde{\mathcal{F}}$, there must exist some open subset U' of U containing x such that the stalks of s within U' all come from a single section s' of $\mathcal{F}$. Then $f(s) \in \mathcal{G}(U')$ has the desired property that $f(s)_y = f_y(s_y)$ for all $y \in U'$.

   Going the other way, let $\tilde{f} : \tilde{\mathcal{F}} \to \mathcal{G}$ be a morphism of sheaves. Recall that by construction, there is a morphism $\mathcal{F} \to \tilde{\mathcal{F}}$ of presheaves which is an isomorphism on stalks. By composing with $\tilde{f}$, we get a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves.

   We still have to check that composing these two maps either way is the identity on either side. The point here is that because $\mathcal{G}$ is a sheaf, objects on either side are determined by their actions on stalks, which are "the same" if we identify stalks of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ in the usual way.

4. We divide the points of $\operatorname{Spec} \mathbb{Z}$ into four subsets.

   - The generic point $\operatorname{Spec} \mathbb{Q}$ has a unique point $\operatorname{Spec} \mathbb{Q}(i)$ in its fiber.
   - The point (2) has a unique point $(1 + i)$ in its fiber.
   - The points (p) for p a prime congruent to 3 modulo 4 each have a unique point (p) in their fibers.
   - The points (p) for p a prime congruent to 1 modulo 4 each have *two* points in their fibres: by Fermat's theorem one can write $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$, and then the ideals $(a + bi)$ and $(a - bi)$ are distinct prime ideals in $\operatorname{Spec} \mathbb{Z}[i]$.

5. These come in two types: ideals of the form $(x - a)$ with $a \in \mathbb{R}$, and ideals of the form $(x^2 + ax + b)$ with $a, b \in \mathbb{R}$ and $x^2 + ax + b$ irreducible. The ideals of the second type correspond to pairs of complex conjugates in $\mathbb{C}$.

6. (a) The sheaf $\mathcal{O}(1)$ is generated by its global sections, which are the homogeneous polynomials of degree 1. These pull back to homogeneous polynomials of degree d, which are sections of $\mathcal{O}(d)$.

   (b) The sheaf $\mathcal{O}(1)$ is generated by its global sections, which are the homogeneous polynomials of degree 1. These pull back to homogeneous polynomials of bidegree $(1, 1)$, which are sections of the external product $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$.

7. This can be done purely in the language of commutative algebra, but we indicate the following proof in order to illustrate ideas from the lectures so far. Suppose first that $X = \mathrm{Spec}(R)$ can be written as the disjoint union of two nonempty open subsets U_1, U_2. Then there is a section $e_1 \in \mathcal{O}_X(X)$ which restricts to 1 on U_1 and 0 on U_2, and a section $e_2 \in \mathcal{O}_X(X)$ which restricts to 1 on U_2 and 0 on U_1. We proved in class that the natural map $R \to \mathcal{O}_X(X)$ is an isomorphism. To check that e_1, e_2 satisfy $e_1 + e_2 = 1$, $e_1^2 = e_1$, $e_2^2 = e_2$, it is enough to check this at the level of sections, which we may do on U_1 and U_2 separately.

   Suppose next that e_1, e_2 are nonzero idempotents which add up to 1. Then $V(e_1), V(e_2)$ are closed subsets of X; they are disjoint because e_1 and e_2 generate the unit ideal, and they cover X because $e_1 e_2 = e_1(1 - e_1) = e_1 - e_1^2 = 0$. Finally, $V(e_1)$ is nonempty: otherwise, e_1 would have to generate the unit ideal, so we could find $f \in R$ with $e_1 f = 1$; but then $e_1 = e_1^2 f = e_1 f = 1$ and so $e_2 = 1 - e_1 = 0$, a contradiction. Similarly, $V(e_2)$ is nonempty, so they form a partition of X into two nonempty closed sets.