# Math 203B (Algebraic Geometry), UCSD, winter 2013
## Solutions for problem set 2

1. (a) We may view $X_i \cap X_j$ as the open subscheme of $X_i$ consisting of those points $x$ for which $f_j$ does not belong to the maximal ideal of the local ring $\mathcal{O}_{X,x} = \mathcal{O}_{X_i,x}$. But if we identify a point $x$ of $X_i = \operatorname{Spec}(A_i)$ with a prime ideal $\mathfrak{p}$ of $A_i$, then $f_j$ is in the maximal ideal of $\mathcal{O}_{X,x}$ if and only if $f_j \in \mathfrak{p}$, so $X_i \cap X_j$ coincides with the distinguished open affine subscheme of $\operatorname{Spec} A_i$ defined by $f_j$. The latter is none other than $\operatorname{Spec} A_i[f_j^{-1}]$.

   (b) Because $\mathcal{O}_X$ is a sheaf, we have an exact sequence

$$0 \to \mathcal{O}_X(X) \to \bigoplus_{i=1}^n \mathcal{O}_X(X_i) \to \bigoplus_{i,j=1}^n \mathcal{O}_X(X_i \cap X_j).$$

   By replacing labels, we get

$$0 \to A \to \bigoplus_{i=1}^n A_i \to \bigoplus_{i,j=1}^n A_{ij}.$$

   Since $A_{f_k}$ is flat over $A$, we get another exact sequence

$$0 \to A_{f_k} \to \bigoplus_{i=1}^n (A_i)_{f_k} \to \bigoplus_{i,j=1}^n (A_{ij})_{f_k}$$

   which we can rewrite using (a) as

$$0 \to A_{f_k} \to \bigoplus_{i=1}^n \mathcal{O}_X(X_i \cap X_k) \to \bigoplus_{i,j=1}^n \mathcal{O}_X(X_i \cap X_j \cap X_k).$$

   But from this it is clear that $A_{f_k} = \mathcal{O}_X(X_k) = A_k$.

   (c) Using (b), we get a ring map $A \to A_{f_i} \cong A_i$ and hence a morphism $X_i \cong \operatorname{Spec}(A_{f_i}) \to \operatorname{Spec}(A)$ of schemes. These maps agree on overlaps, so they define a morphism $X \to \operatorname{Spec}(A)$ of schemes. To see that this is an isomorphism, it suffices to check locally on $A$. But $f_1, \dots, f_n$ generate the unit ideal in $A$, so the distinguished opens $D(f_i)$ form a cover, and the restriction to $D(f_i)$ is the isomorphism $X_i \cong \operatorname{Spec}(A_i)$.

2. We prove the result more generally allowing $C_1, C_2$ to be reducible as long as they have no common component. Let $V$ be the space of homogeneous cubic polynomials in three variables over our given field; it is of dimension 10. Let $Q_1, Q_2$ be polynomials cutting out $C_1, C_2$; since $C_1, C_2$ are distinct, $Q_1$ and $Q_2$ are not scalar multiples of each other. For $i = 0, \dots, 9$, let $V_i$ be the subspace of polynomials which vanish at $P_1, \dots, P_8$. It is clear that $\dim(V_i) \geq \dim(V_{i-1}) - 1$ for $i = 1, \dots, 9$. We will prove that equality holds

for $i = 1, \ldots, 8$; this will then imply that $\dim(V_8) = \dim(V_9) = 2$, from which the claim will follow.

To prove the desired equality, it is enough to check that $V_i \neq V_{i-1}$ for $i = 1, \ldots, 8$. That is, we must produce elements of $V_i - V_{i-1}$ for $i = 1, \ldots, 8$, or in other words, cubic curves (possibly reducible) passing through $P_1, \ldots, P_{i-1}$ but not $P_i$. We do this as follows.

- $i = 1$ Any three lines not passing through $P_1$.
- $i = 2$ Any three lines passing through $P_1$ but not $P_2$.
- $i = 3$ Any line not passing through $P_3$ plus any conic passing through $P_1, P_2$ but not $P_3$.
- $i = 4$ Any line not passing through $P_4$ plus any conic passing through $P_1, P_2, P_3$ but not $P_4$.
- $i = 5$ Any line not passing through $P_5$ plus any conic passing through $P_1, P_2, P_3, P_4$ but not $P_5$.
- $i = 6$ Any line passing through $P_1$ but not $P_6$ plus any conic passing through $P_2, P_3, P_4, P_5$ but not $P_6$.
- $i = 7$ Note that no four of our points can be collinear by Bézout's theorem. We can thus find $P_j$ such that the line through $P_1$ and $P_j$ fails to pass through $P_7$. Then add a conic through the other four of $P_1, \ldots, P_6$ not passing through $P_7$.
- $i = 8$ Again by Bézout, no more than six of our points can lie on a conic. If both the conic through $P_1, \ldots, P_5$ and the line through $P_6, P_7$ fail to pass through $P_8$, take the union of these. If this fails because the conic passes through $P_8$, then there must exist an index $j \in \{1, \ldots, 5\}$ such that the line through $P_7$ and $P_8$ fails to pass through $P_j$ (in fact only one index can fail); take the line through $P_j, P_7$ and the conic through the other five points. If instead the line through $P_6, P_7$ passes through $P_8$, then there must exist an index $j \in \{6, 7\}$ such that the conic through $P_1, \ldots, P_4, P_8$ fails to pass through $P_j$ (again because only one index can fail); take the conic through $P_1, \ldots, P_4, P_j$ and the line through the other two points.

3. Pick any point $O$ of $C$ at which the tangent line is triply tangent (it turns out there are exactly 9 such points but we don't need to know this). We first define the inverse map: $-P$ is the third intersection of $C$ with the line $OP$. We then declare the sum $P + Q$ to be the inverse of the third intersection of $C$ with the line $PQ$; this operation is clearly symmetric and has identity $O$. To check associativity, let $P, Q, R$ be any three points. Then the following sets of points are collinear:

   $$P, Q, -P-Q; -P-Q, O, P+Q; P+Q, R, -(P+Q)-R; Q, R, -Q-R; -Q-R, O, Q+R; Q+R, P, -P-$$

   The first three lines form one reducible cubic, while the second three form another one. Since these share eight intersection points, they must also share the ninth, so $-(P + Q) - R = -P - (Q + R)$. This proves the desired associativity.

4. We first construct the sequence in question with $\mathcal{L} = \mathcal{O}_X$. Recall that by definition, $\mathcal{O}_X(-P)$ is the sheaf whose sections over an open subset $U$ are the rational functions with no poles in $U$ and, in case $P \in U$, at least a single zero at $P$. This is obviously a subsheaf of $\mathcal{O}_X$, and the restrictions of $\mathcal{O}_X$ and $\mathcal{O}_X(-P)$ to $X - \{P\}$ coincide. So the quotient $\mathcal{O}_X/\mathcal{O}_X(-P)$ is supported at $P$; on any open affine subscheme $U$ of $X$ containing $P$, it is the quasicoherent sheaf corresponding to the module over $\mathcal{O}_X(U)$ given by quotienting by the prime ideal corresponding to $P$. This is the desired sequence.

   To get the sequence in general, note that $\mathcal{L}$ is locally free, so tensoring with it is an exact operation. Then note that $\mathcal{L}$ can be trivialized on some neighborhood of $P$, so $\mathcal{L} \otimes_{\mathcal{O}} k_P \cong k_P$.

5. It suffices to check the claim when $Y$ is affine; in this case, $X$ is itself quasicompact. (Namely, $Y$ is covered by opens whose inverse images are quasicompact, but only finitely many are needed because $Y$ is also quasicompact.) Pick open affine subsets $U_1, \dots, U_n$ which cover $X$. Because $f$ is quasiseparated, for any $i, j$, the space $X \times_{X \times_Y X} U_i \times_Y U_j$ is quasicompact, but this space is none other than $U_i \cap U_j$. We can thus choose finitely many open affine subsets $V_{ijk}$ of $X$ that cover $U_i \cap U_j$. Let $\mathcal{F}$ be a quasicoherent sheaf on $X$; its pushforward is then the sheaf associated to the module which is the kernel of the map
$$\bigoplus_{i=1}^{n} \mathcal{F}(U_i) \to \bigoplus_{i,j=1}^{n} \bigoplus_{k} \mathcal{F}(V_{ijk}).$$

6. The last map in the sequence
$$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j}$$
   was defined to take $(s_i)_{i=1}^n$ to $(s_i - s_j)_{i,j=1}^n$ (where the restriction maps have been left implicit). The last map in the extended sequence
$$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j} \to \bigoplus_{i,j,k=1}^{n} M_{f_i f_j f_k} \to \cdots$$
   can be taken to send $(s_{ij})_{i,j=1}^n$ to $(s_{ij} - s_{ik} + s_{jk})_{i,j,k=1}^n$.

7. The only thing that needs to be checked is surjectivity of $\mathcal{G}(X) \to \mathcal{H}(X)$. Choose $s \in \mathcal{H}(X)$. For some open affine cover $\{U_i\}_{i \in I}$ of $X$, we can lift $s|_{U_i}$ to some section $t_i \in \mathcal{G}(U_i)$. By assumption, there exists an open immersion $j : Y \to X$ such that $Y$ can be written as an affine scheme $\mathrm{Spec}(R)$ and $\mathcal{F}$ is the pushforward of $\tilde{M}$ for some $R$-module $M$. By making the $U_i$ small enough, we may ensure that each intersection $U_i \cap Y$ is a distinguished open subset $D(f_i)$ of $\mathrm{Spec}(R)$.

For each pair $i, j$, the difference $t_i - t_j$ is a section in $\mathcal{G}(U_i \cap U_j)$ which maps to zero in $\mathcal{H}(U_i \cap U_j)$, so we can view it as a section in $\mathcal{F}(U_i \cap U_j)$ and hence as an element $m_{ij} \in M_{f_i f_j}$. Note that for all $i, j, k$, we have $m_{ij} + m_{jk} = m_{ik}$ because the corresponding equality in $\mathcal{G}(U_i \cap U_j \cap U_k)$ is the triviality $(t_i - t_j) + (t_j - t_k) = (t_i - t_k)$.

By the exact sequence in the previous exercise, we can find elements $m_i \in M_{f_i}$ such that $m_i - m_j = m_{ij}$. Now $t_i - m_i$ is another section of $\mathcal{G}(U_i)$ lifting $s$, but the differences $(t_i - m_i) - (t_j - m_j)$ vanish in $\mathcal{G}(U_i \cap U_j)$. So these patch together to give a section $t \in \mathcal{G}(X)$ lifting $s$.