Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 2 (due *Friday*, January 25)

This problem set is due later than usual because there will be no lectures on Friday, January 18 or Monday, January 21. Solve the following problems, and turn in the solutions to *four* of them.

1. Let X be a scheme and put $A = \mathcal{O}_X(X)$. Let $f_1, \dots, f_n \in A$ be elements which generate the unit ideal. For $i = 1, \dots, n$, let X_i be the open subscheme of X consisting of those points x for which f does not belong to the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Suppose that X_i is affine for $i = 1, \dots, n$, and put $A_i = \mathcal{O}_X(X_i)$.
   (a) For $i, j = 1, \dots, n$, prove that the open subscheme $X_i \cap X_j$ of X is isomorphic to $\operatorname{Spec}(A_i[f_j^{-1}])$.
   (a) Prove that the natural map $A_{f_i} \to A_i$ is an isomorphism. Hint: start with an exact sequence
   $$0 \to A \to \bigoplus_{i=1}^{n} A_i \to \bigoplus_{i,j=1}^{n} A_{ij}$$
   for $A_{ij} = \mathcal{O}_X(X_i \cap X_j)$, then invert f_i.
   (b) Prove that X is isomorphic to $\operatorname{Spec}(A)$.

2. Let C_1, C_2 be distinct smooth cubic curves in $\mathbb{P}^2$ over a field, and suppose that they meet in nine distinct points $P_1, \dots, P_9$. Prove that any (possibly reducible) cubic curve passing through $P_1, \dots, P_8$ also passes through P_9. Hint: show that the vanishing at $P_1, \dots, P_8$ impose linearly independent conditions on a cubic polynomial (e.g., using reducible cubics) and then count dimensions.

3. Let C be a smooth cubic curve in $\mathbb{P}^2$ over a field. Prove that there exists a group structure on C with the property that any three collinear points on C sum to zero. You may use the previous exercise whether or not you are submitting it.

4. Let X be a smooth curve over a field k, let $\mathcal{L}$ be a locally free sheaf of rank 1 over X, and let P be a point of X. Construct an exact sequence
   $$0 \to \mathcal{L}(-P) \to \mathcal{L} \to k_P \to 0.$$

5. Read the proof of Proposition 7.2.9 in the Gathmann notes, then prove the following generalization. A morphism $f : X \to Y$ of schemes is *quasicompact* if Y can be covered by open affine subspaces whose inverse images are quasicompact. The morphism f is *quasiseparated* if the diagonal morphism $X \to X \times_Y X$ is quasicompact. Prove that if f is quasicompact and quasiseparated, then the pushforward of any quasicoherent sheaf on X is a quasicoherent sheaf on Y.

6. Here's a fact we'll use soon in the construction of sheaf cohomology. Let M be a module over a ring R. Let $f_1, \ldots, f_n$ be elements of R which generate the unit ideal. In class, we constructed an exact sequence

$$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j}.$$

Show that this extends to an exact sequence

$$0 \to M \to \bigoplus_{i=1}^{n} M_{f_i} \to \bigoplus_{i,j=1}^{n} M_{f_i f_j} \to \bigoplus_{i,j,k=1}^{n} M_{f_i f_j f_k} \to \cdots,$$

where the definition of the additional terms and maps is left for you to figure out. (It might help to try the case $n = 3$ first, then look for the general pattern.)

7. Let X be a scheme. Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence of quasicoherent sheaves on X (so in particular the maps are $\mathcal{O}_X$-linear). Suppose that there exists an open immersion $j : Y \to X$ such that Y is affine and $\mathcal{F}$ is the pushforward of a quasicoherent sheaf on Y. Prove that the sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

of global sections is again exact. We will see later how to generalize this statement using sheaf cohomology.