Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 3 (due Wednesday, January 30)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent sheaf on X which is locally finitely generated. Prove that the function $x \mapsto \dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ is upper semicontinuous.

2. Let R be a ring. Give a careful proof that $\Omega_{R[x_1,\dots,x_n]/R}$ is freely generated by $dx_1, \dots, dx_n$. The part that needs care is to show that there are no relations among these elements; you may wish to use the universal property of $\Omega_{R[x_1,\dots,x_n]/R}$.

3. Let R be any ring. For f in the formal Laurent series ring $R((T))$, the *residue* of the formal differential form $f\,dT$ is defined as the coefficient of $T^{-1}\,dT$. Prove that the residue of $f\,dT$ is invariant under any substitution of the from $T \mapsto a_1T + a_2T^2 + \cdots$ with $a_1 \in k^\times$ and $a_2, a_3, \dots \in k$. Hint: reduce this to a collection of polynomial identities over $\mathbb{Z}$, which can then be checked using the usual residue theorem from complex analysis.

4. Let k be an algebraically closed field. Show that there is a unique way to assign a *residue* to each meromorphic differential ω on $\mathbb{P}^1_k$ at each point P of $\mathbb{P}^1_k$ satisfying the following conditions.

   (i) For $P = 0$, the residue is computed by writing $\omega = f dT$ and taking the residue of $f\,dT$ as in the previous exercise (i.e., the coefficient of $T^{-1}\,DT$).

   (ii) If L is a linear fractional transformation, then the residue of ω at $L(P)$ is the same as the residue of $L^*(\omega)$ at P.

   For this, you may invoke the previous exercise whether or not you submit it.

5. Let k be an algebraically closed field. Prove the *residue theorem* for $\mathbb{P}^1_k$: for any meromorphic differential ω on $\mathbb{P}^1_k$, the sum of the residues of ω over all points of $\mathbb{P}^1_k$ (as defined in the previous exercise) is equal to 0. Hint: one possible approach is again reduction to the case $k = \mathbb{C}$.

6. Let k be an algebraically closed field. Let C be a smooth projective curve over k. Let P be a closed point of C. Using Riemann-Roch, prove that there exists a meromorphic function on C with no poles at any points other than P.

7. Let k be an algebraically closed field. Let C be a smooth projective curve of genus 2 over k. Using the Riemann-Roch theorem, prove the following.

   (i) There exists a degree 2 map from C to $\mathbb{P}^1_k$.

   (ii) There is an embedding of C in $\mathbb{P}^3_k$ of degree 5.