Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 5

1. By hypothesis, there exist $f_1, \dots, f_n \in R$ generating the unit ideal such that $\tilde{M}$ is finitely generated over $D(f_i)$; that is, the module M_{f_i} over R_{f_i} is finitely generated. Since every element of M_{f_i} can be written as an element of M divided by a power of f_i, we can find elements $m_{i,1}, \dots, m_{i,n_i}$ of M whose images in M_{f_i} generate M_{f_i} as an R_{f_i}-module. Let N be the submodule of M generated by all of the $m_{i,j}$; then the map $\tilde{N} \to \tilde{M}$ is an isomorphism on stalks, so we must have $N = M$. Therefore M is finitely generated.

2. Let a, b and c, d be homogeneous coordinates on two copies of $\mathbb{P}^1$. Let $P(x)$ be a polynomial over k of degree $2g+2$ with no repeated roots, and consider the subspace of $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by $d^2 = c^2 b^{2g+2} P(a/b)$. This is a curve C with a 2-to-1 map to $\mathbb{P}^1$ ramified at $2g+2$ points, each with ramification index 2, so Riemann-Hurwitz says that $2g(C) - 2 = 2 \cdot (-2) + 2g + 2$, and so $g(C) = g$.

3. (a) Since we are in characteristic p, we have $dy = -dx$, so we get no ramification anywhere where dx generates $\Omega_{\mathbb{P}^1}$. Hence we get no ramification away from $x = \infty$.

   (b) Let's change coordinates by setting $t = 1/x$, so we can work at $t = 0$. Then

   $$df^*(y^{-1}) = d((x^p - x)^{-1}) = \frac{-dx}{(x^p - x)^2} = \frac{dt}{t^2(t^{-p} - t^{-1})^2} = \frac{t^{2p-2}\,dt}{(1 - t^{p-1})^2},$$

   so the ramification index is $2p - 1$.

   (c) The naïve ramification index is p because the degree of the map is p and there is only one point in the fiber.

   (d) Riemann-Hurwitz here says $-2 = p(-2) + (2p - 2)$, which checks.

4. For n sufficiently large, $H^0(C, \mathcal{O}(n))$ is isomorphic to $H^0(\mathbb{P}^2, \mathcal{O}(n))/H^0(\mathbb{P}^2, \mathcal{O}(n-d))$ and so has dimension

$$\binom{n+2}{2} - \binom{n+2-d}{2} = dn + \frac{d^2 - 3d}{2} = dn - 1 + g.$$

5. (a) By hypothesis, there exist homogeneous polynomials $f_1, \dots, f_m$ such that the distinguished open sets $D(f_i)$ cover $\mathbb{P}^d_R$ and $\mathcal{F}$ is finitely generated on $D(f_i)$. By raising the f_i to suitable powers, we may force them all to be of a single degree d. Then for each $s \in H^0(D(f_i), \mathcal{F})$, for n sufficiently large $f^n s$ lifts to a global section of $\mathcal{F}(dn)$. Putting these sections together gives a set of generators as in the first exercise. This almost proves the claim, except that we are only getting multiples of d; but if the claim holds for some n, then it also holds for $n+1$ because we can just multiply the generators of $\mathcal{F}(n)$ by $x_0, \dots, x_d$ in turn to get generators of $\mathcal{F}(n+1)$.

(b) Apply (a) to $j_*\mathcal{F}$.

6. (a) Using Zorn's lemma (or transfinite induction or your favorite other equivalent of the axiom of choice), it suffices to check the injectivity property for an injection $B \to C$ where C/B is generated by a single element c. If C/B is finite of order n, then by hypothesis we can divide the image of nc in A by n and send c there. If C/B is infinite, we can send c wherever we like (to 0, for example).

   (b) We are saying that if $a \in A$ is nonzero, then there exists $b \in \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ such that $b(a)$ is nonzero. To check this, note that the subgroup of A generated by a always admits a nonzero map to $\mathbb{Q}/\mathbb{Z}$, then apply (a) to extend this map to all of A.

   (c) Since $(\mathcal{F}_x)'$ is injective and $\mathcal{G}_x \to \mathcal{H}_x$ is an injection, the morphism $\mathcal{G}_x \to (\mathcal{F}_x)'$ extends to a morphism $\mathcal{H}_x \to (\mathcal{F}_x)'$. For each U, we map $\mathcal{H}(U)$ to $\mathcal{F}'(U)$ by injecting $\mathcal{H}(U)$ into $\prod_{x\in U} \mathcal{H}_x$ and then mapping term-by-term to $\prod_{x\in U}(\mathcal{F}_x)' = \mathcal{F}'(U)$.