Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 5 (due Wednesday, February 13)

Solve the following problems, and turn in the solutions to four of them.

1. Let \( R \) be a commutative ring. Let \( M \) be an \( R \)-module such that the quasicoherent sheaf \( \tilde{M} \) on \( \text{Spec}(R) \) is locally finitely generated. Prove that \( M \) is itself finitely generated. (If you find yourself needing to assume that \( R \) is noetherian, you are probably not doing this correctly.) For this reason, from now on I will talk about finitely generated quasicoherent sheaves, leaving out the word “locally”.

2. Let \( k \) be an algebraically closed field of characteristic not equal to 2. Prove that for every nonnegative integer \( g \), there exists a curve over \( k \) of genus \( g \). Hint: use hyperelliptic curves and Riemann-Hurwitz. (One can make the argument work in characteristic 2 also, but you needn’t do so here.)

3. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( f \) be the map of degree from \( X = \mathbb{P}^1 \) (the \( x \)-line) to \( Y = \mathbb{P}^1 \) (the \( y \)-line) for which \( f^*(y) = x^p - x \).
   (a) Prove that \( f \) has no ramification over any of the points of \( \mathbb{A}^1 \).
   (b) Compute the ramification number at \( x = \infty \) by computing the order of \( df^*(y^{-1}) \) at \( x = \infty \) and then adding 1. Notice that it is greater than \( p - 1 \)!
   (c) Compute the naïve ramification number at \( x = \infty \) by computing the lengths of components of the scheme \( X \times_Y \text{Spec}(k(\infty)) \). Notice that it does not match (b)!
   (d) Write out all of the terms of the Riemann-Hurwitz formula for this map. Of course the formula had better hold in this case!

4. In this exercise, we will complete the proof of Riemann-Roch in the special case of smooth plane curves. Let \( k \) be an algebraically closed field. Let \( C \) be a smooth curve of degree \( d \) in \( \mathbb{P}^2 \). Recall that we (mostly) proved on a previous problem set that \( g(C) = \frac{(d - 1)(d - 2)}{2} \).
   (a) Prove that \( h^0(C, \mathcal{O}(n)) = dn - 1 + g \) for \( n \) sufficiently large. (That is, the quantity \( g' \) mentioned in class equals \( g \).)
   (b) Prove that \( \text{deg}(K_C) = 2g - 2 \) using the isomorphism \( \omega_C \cong \mathcal{O}(d - 3) \).

5. Let \( R \) be a ring.
   (a) Let \( \mathcal{F} \) be a locally finitely generated quasicoherent sheaf on the projective space \( \mathbb{P}^d_R \). Prove that for every sufficiently large integer \( n \), there exist finitely many elements of \( H^0(\mathbb{P}^d_R, \mathcal{F}(n)) \) which generate \( \mathcal{F}(n) \). Hint: if \( s \) is a section in \( H^0(D(f), \mathcal{F}) \), then for \( n \) sufficiently large \( f^n s \) extends to a section in \( H^0(\mathbb{P}^d_R, \mathcal{F}(n \text{deg}(f))) \).
(b) Let \( j : X \to \mathbb{P}^d_R \) be a closed immersion, and use it to define the twisting sheaves \( \mathcal{O}(n) \) for \( n \in \mathbb{Z} \). Let \( \mathcal{F} \) be a (locally) finitely generated quasicoherent sheaf on \( X \). Prove that for every sufficiently large integer \( n \), the \( R \)-module \( H^0(X, \mathcal{F}(n)) \) is finitely generated. Hint: push forward to reduce to (a).

6. In this exercise, we address one of the foundational issues in the general construction of sheaf cohomology.

(a) An abelian group \( A \) is \textit{injective} if for any injection \( B \to C \), every morphism \( B \to A \) can be extended (not necessarily uniquely) to a morphism \( C \to A \). Prove that any \textit{divisible} abelian group (i.e., one for which for each positive integer \( n \) the multiplication-by-\( n \) map is surjective) is injective. Hint: using Zorn’s lemma, it suffices to check the injectivity property for an injection \( B \to C \) where \( C/B \) is generated by a single element. Or Google for “Baer’s criterion.”

(b) For any abelian group \( A \), put
\[
A' = \prod_{b \in \text{Hom}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}.
\]
Prove that the evaluation map \( A \to A' \), which takes \( a \) to the tuple whose \( b \)-component is \( b(a) \), is injective. Hint: reduce this to (a).

(c) Let \( X \) be a topological space. Let \( \mathcal{F} \) be a sheaf of abelian groups on \( X \). Define the sheaf \( \mathcal{F}' \) by
\[
\mathcal{F}'(U) = \prod_{x \in U} (\mathcal{F}_x)',
\]
so that there is an obvious injection \( \mathcal{F} \to \mathcal{F}' \). Prove that \( \mathcal{F}' \) is an \textit{injective sheaf}: for any injective morphism \( \mathcal{G} \to \mathcal{H} \) of sheaves of abelian groups on \( X \), every morphism \( \mathcal{G} \to \mathcal{F}' \) can be extended (not necessarily uniquely) to a morphism \( \mathcal{H} \to \mathcal{F}' \). Consequently, the category of sheaves of abelian groups on \( X \) has \textit{enough injectives}. 

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