Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 5 (due Wednesday, February 13)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let R be a commutative ring. Let M be an R-module such that the quasicoherent sheaf $\tilde{M}$ on $\mathrm{Spec}(R)$ is locally finitely generated. Prove that M is itself finitely generated. (If you find yourself needing to assume that R is noetherian, you are probably not doing this correctly.) For this reason, from now on I will talk about *finitely generated* quasicoherent sheaves, leaving out the word "locally".

2. Let k be an algebraically closed field of characteristic not equal to 2. Prove that for every nonnegative integer g, there exists a curve over k of genus g. Hint: use hyperelliptic curves and Riemann-Hurwitz. (One can make the argument work in characteristic 2 also, but you needn't do so here.)

3. Let k be an algebraically closed field of characteristic $p > 0$. Let f be the map of degree from $X = \mathbb{P}^1$ (the x-line) to $Y = \mathbb{P}^1$ (the y-line) for which $f^*(y) = x^p - x$.

   (a) Prove that f has no ramification over any of the points of $\mathbb{A}^1$.

   (b) Compute the ramification number at $x = \infty$ by computing the order of $df^*(y^{-1})$ at $x = \infty$ and then adding 1. Notice that it is greater than $p - 1$!

   (c) Compute the naïve ramification number at $x = \infty$ by computing the lengths of components of the scheme $X \times_Y \mathrm{Spec}(\kappa(\infty))$. Notice that it does not match (b)!

   (d) Write out all of the terms of the Riemann-Hurwitz formula for this map. Of course the formula had better hold in this case!

4. In this exercise, we will complete the proof of Riemann-Roch in the special case of smooth plane curves. Let k be an algebraically closed field. Let C be a smooth curve of degree d in $\mathbb{P}^2$. Recall that we (mostly) proved on a previous problem set that $g(C) = (d-1)(d-2)/2$.

   (a) Prove that $h^0(C, \mathcal{O}(n)) = dn - 1 + g$ for n sufficiently large. (That is, the quantity g' mentioned in class equals g.)

   (b) Prove that $\deg(K_X) = 2g - 2$ using the isomorphism $\omega_C \cong \mathcal{O}(d-3)$.

5. Let R be a ring.

   (a) Let $\mathcal{F}$ be a locally finitely generated quasicoherent sheaf on the projective space $\mathbb{P}^d_R$. Prove that for every sufficiently large integer n, there exist finitely many elements of $H^0(\mathbb{P}^d_R, \mathcal{F}(n))$ which generate $\mathcal{F}(n)$. Hint: if s is a section in $H^0(D(f), \mathcal{F})$, then for n sufficiently large $f^n s$ extends to a section in $H^0(\mathbb{P}^d_R, \mathcal{F}(n \deg(f)))$.

(b) Let $j : X \to \mathbb{P}^d_R$ be a closed immersion, and use it to define the twisting sheaves $\mathcal{O}(n)$ for $n \in \mathbb{Z}$. Let $\mathcal{F}$ be a (locally) finitely generated quasicoherent sheaf on X. Prove that for every sufficiently large integer n, the R-module $H^0(X, \mathcal{F}(n))$ is finitely generated. Hint: push forward to reduce to (a).

6. In this exercise, we address one of the foundational issues in the general construction of sheaf cohomology.

   (a) An abelian group A is *injective* if for any injection $B \to C$, every morphism $B \to A$ can be extended (not necessarily uniquely) to a morphism $C \to A$. Prove that any *divisible* abelian group (i.e., one for which for each positive integer n the multiplication-by-n map is surjective) is injective. Hint: using Zorn's lemma, it suffices to check the injectivity property for an injection $B \to C$ where C/B is generated by a single element. Or Google for "Baer's criterion."

   (b) For any abelian group A, put

   $$A' = \prod_{b \in \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}.$$

   Prove that the evaluation map $A \to A'$, which takes a to the tuple whose b-component is $b(a)$, is injective. Hint: reduce this to (a).

   (c) Let X be a topological space. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Define the sheaf $\mathcal{F}'$ by

   $$\mathcal{F}'(U) = \prod_{x \in U} (\mathcal{F}_x)',$$

   so that there is an obvious injection $\mathcal{F} \to \mathcal{F}'$. Prove that $\mathcal{F}'$ is an *injective sheaf*: for any injective morphism $\mathcal{G} \to \mathcal{H}$ of sheaves of abelian groups on X, every morphism $\mathcal{G} \to \mathcal{F}'$ can be extended (not necessarily uniquely) to a morphism $\mathcal{H} \to \mathcal{F}'$. Consequently, the category of sheaves of abelian groups on X *has enough injectives.*