Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 6

1. There are enough proofs of the snake lemma posted online that it hardly seems necessary to include one here. See for example `http://planetmath.org/?op=getobj&id=5578&from=objects`.

2. (a) Both the hypothesis and the conclusion are preserved by tensoring with $\mathcal{O}(n)$: that is, we can tensor with $\mathcal{O}(n)$, make the splitting, then tensor with $\mathcal{O}(-n)$. So we may as well assume that $n_2 = 0$ and $n_1 \geq 0$. In this case, the long exact sequence in cohomology reads

$$H^0(\mathbb{P}^1_k, \mathcal{F}) \to H^0(\mathbb{P}^1_k, \mathcal{O}) \to H^1(\mathbb{P}^1_k, \mathcal{O}(n_1))$$

but the last term vanishes by the calculations made in class. So any nonzero section of $\mathcal{O}$ lifts to $\mathcal{F}$, which gives the splitting.

(b) Again, we can tensor with $\mathcal{O}(n)$, produce a new exact sequence, then tensor with $\mathcal{O}(-n)$ at the end. So we may assume $n_1 = -1$ and $n_2 \geq 0$. This time, the long exact sequence in cohomology reads

$$H^0(\mathbb{P}^1_k, \mathcal{F}) \to H^0(\mathbb{P}^1_k, \mathcal{O}(n_2)) \to H^1(\mathbb{P}^1_k, \mathcal{O}(-1))$$

and again the last term is zero, so $\mathcal{F}$ must admit a nonzero section. Use some such section to form the exact sequence

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{G} \to 0.$$

Here we should be careful: the quotient $\mathcal{G}$ is of rank 1 but might have some torsion. But since we are in dimension 1 (or if you prefer, because the rings of sections are all PIDs), the quotient $\mathcal{G}'$ of $\mathcal{G}$ by its torsion is free of rank 1. So now I have a new exact sequence

$$0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{G}' \to 0$$

in which $\mathcal{G}'$ and $\mathcal{H}$ are both locally free of rank 1 and $\mathcal{H}$ admits a nonzero section (since it contains the copy of $\mathcal{O}$ from the previous exact sequence). Since every line bundle on $\mathbb{P}^1_k$ is an $\mathcal{O}(n)$ for some n, we can rewrite this exact sequence as

$$0 \to \mathcal{O}(n_1') \to \mathcal{F} \to \mathcal{O}(n_2') \to 0$$

for some $n_1', n_2' \in \mathbb{Z}$. Since $\mathcal{O}(n_1')$ admits a nonzero section, we must have $n_1' \geq 0$; since

$$\mathcal{O}(n_1' + n_2') \cong \wedge^2 \mathcal{F} \cong \mathcal{O}(n_1 + n_2),$$

we must have $n_2' = n_2 - (n_1' - n_1)$. This proves the claim.

3. (a) By a previous homework, for n sufficiently large, $\mathcal{F}(n)$ is generated by global sections, so we can find an exact sequence

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{G} \to 0.$$

By replacing $\mathcal{G}$ with its torsion-free quotient $\mathcal{G}'$ as in the previous exercise, we can turn this into a new exact sequence

$$0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{G}' \to 0$$

in which $\mathcal{H}$ is locally free of rank 1 (and hence isomorphic to some $\mathcal{O}(n_1)$) and $\mathcal{G}'$ is locally free of rank $d-1$. We may thus continue by induction.

(b) For any filtration as in (a), we have

$$\wedge^d \mathcal{F} \cong \bigotimes_{i=1}^{d} \mathcal{F}_i/\mathcal{F}_{i-1},$$

so the sum $n_1 + \cdots + n_d$ is independent of the filtration. Suppose there exists an index i for which $n_i \leq n_{i-1} - 1$. Apply the previous exercise to $\mathcal{F}_i/\mathcal{F}_{i-2}$ to get a new filtration

$$0 \to \mathcal{O}(n_{i-1} + c) \to \mathcal{F}_i/\mathcal{F}_{i-2} \to \mathcal{O}(n_i - c) \to 0$$

for some positive integer c. If we change the original filtration by replacing $\mathcal{F}_{i-1}$ with the inverse image of $\mathcal{O}(n_{i-1}+c)$ under the map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-2}$, we get a new filtration with the values n_{i-1}, n_i replaced by $n_{i-1} + c, n_i - c$.

Now the plan is to perform this operation until we no longer can; for this to work, we must check that the process is forced to terminate after some finite number of steps. One way to quantify this is to look at the quantities

$$n_1, n_1 + n_2, \cdots, n_1 + \cdots + n_d.$$

At each step, one of these numbers gets bigger and the rest remain unchanged. However, n_1 cannot grow without bound, as otherwise we would have

$$\dim_k H^0(\mathbb{P}^1_k, \mathcal{F}) \geq \dim_k H^0(\mathbb{P}^1_k, \mathcal{O}(n)) = n + 1$$

for arbitrarily large n, contradicting the fact that $H^0(\mathbb{P}^1_k, \mathcal{F})$ is finite-dimensional over k. So at some point n_1 must stop growing. After that point, $\mathcal{F}_1$ remains fixed forever, so we can apply the same argument to $\mathcal{F}/\mathcal{F}_1$ to see that at some point $n_1 + n_2$ must stop growing, and so on.

Once the process stops, we must have $n_1 \geq n_2 \geq \cdots \geq n_d$. It will complete the proof to show that under this condition, the whole filtration splits into a direct sum. We check this by induction on d, there being nothing to check for $d = 1$. If $d \geq 2$, then by the previous problem the exact sequence

$$0 \to \mathcal{O}(n_{d-1}) \to \mathcal{F}/\mathcal{F}_{d-2} \to \mathcal{O}(n_d) \to 0$$

splits. Choose a splitting $\mathcal{F}/\mathcal{F}_{d-2} \cong \mathcal{O}(n_{d-1}) \oplus \mathcal{O}(n_d)$ and let $\mathcal{G}$ be the inverse image of $\mathcal{O}(n_d)$ under the surjection $\mathcal{F} \to \mathcal{F}/\mathcal{F}_{d-2}$. Then $\mathcal{G}$ is of rank $d-1$ and admits the filtration

$$0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{d-2} \subseteq \mathcal{G}$$

where the successive quotients are $\mathcal{O}(n_1), \dots, \mathcal{O}(n_{d-2}), \mathcal{O}(n_d)$. Since the degrees are again nondecreasing, we can use the induction hypothesis to split this; that is, we get an isomorphism $\mathcal{G} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_{d-2}) \oplus \mathcal{O}(n_d)$ and a short exact sequence

$$0 \to \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_{d-2}) \oplus \mathcal{O}(n_d) \to \mathcal{F} \to \mathcal{O}(n_{d-1}) \to 0.$$

All that remains is to split this last sequence; for this, we may assume $n_{d-1} = 0$, so that the task at hand is to show that

$$H^0(\mathbb{P}^1_k, \mathcal{F}) \to H^0(\mathbb{P}^1_k, \mathcal{O})$$

is surjective. The next term in the long exact sequence is

$$\begin{aligned} &H^1(\mathbb{P}^1_k, \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_{d-2}) \oplus \mathcal{O}(n_d)) \\ &= H^1(\mathbb{P}^1_k, \mathcal{O}(n_1)) \oplus \cdots \oplus H^1(\mathbb{P}^1_k, \mathcal{O}(n_{d-2})) \oplus H^1(\mathbb{P}^1_k, \mathcal{O}(n_d)). \end{aligned}$$

Since $n_1 \geq \cdots \geq n_{d-2} \geq n_{d-1} = 0$, all of the terms $H^1(\mathbb{P}^1_k, \mathcal{O}(n_i))$ for $i = 1, \dots, d-2$ vanish, so the term in question is just $H^1(\mathbb{P}^1_k, \mathcal{O}(n_d))$. But the connecting map $H^0(\mathbb{P}^1_k, \mathcal{O}) \to H^1(\mathbb{P}^1_k, \mathcal{O}(n_d))$ is the same one that comes from the exact sequence

$$0 \to \mathcal{O}(n_{d-1}) \to \mathcal{F}/\mathcal{F}_{d-2} \to \mathcal{O}(n_d) \to 0$$

which we already know is split. So this connecting map must be zero, giving the desired surjectivity.

4. (a) Let $f : X \to Y = \operatorname{Spec}(A)$ be a closed immersion. The kernel of $f^\sharp$ is a quasicoherent subsheaf of the structure sheaf on $\mathcal{O}_X$, so corresponds to an A-submodule I of A. But then $X \cong \operatorname{Spec}(A/I)$.

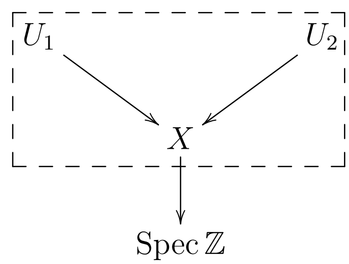
   (b) The definition of a closed immersion is local on the target (i.e., both the topological part and the sheaf-theoretic part can be checked locally on the target), so we may assume that both X and Z are affine. By (a), Y is also affine, so we may view $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(A/I)$, $Z = \operatorname{Spec}(B)$. Let $f : A \to B$ be the ring homomorphism corresponding to $Z \to X$; then $Y \times_X Z \to Z$ corresponds to the map $B \to B/f(I)B$ and so is a closed immersion.

5. (a) Put $Y = X \times_{\operatorname{Spec}\mathbb{Z}} X$. Recall first that the open subscheme $U_1 \cap U_2$ of X can be identified with $U_1 \times_X U_2$: there are obvious open immersions $U_1 \cap U_2 \to U_1$, $U_1 \cap U_2 \to U_2$ such that the compositions $U_1 \cap U_2 \to U_1 \to X$ and $U_1 \cap U_2 \to U_2 \to X$ coincide, so there is a map $U_1 \cap U_2 \to U_1 \times_X U_2$. To check that this is an isomorphism, it is sufficient to check in a neighborhood of a point of $U_1 \times_X U_2$. That point maps to specific points of U_1, U_2, X and we may work locally around all of those, so we may assume $X = \operatorname{Spec} A$ is affine and $U_1 = D(f_1), U_2 = D(f_2)$ are distinguished open subsets. But then $U_1 \cap U_2 = D(f_1 f_2)$ and $U_1 \times_X U_2 = \operatorname{Spec}(A_{f_1} \otimes_A A_{f_2})$, so the identification is clear.

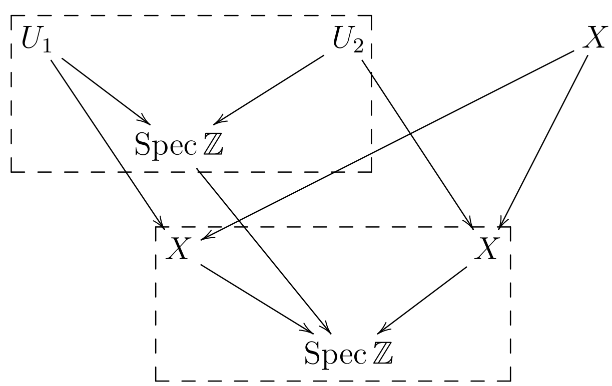
Now we change the identification by producing an isomorphism

$$U_1 \times_X U_2 \cong (U_1 \times_{\operatorname{Spec}\mathbb{Z}} U_2) \times_Y X.$$

This is most easily explained by picture: the left side is the universal object mapping into the diagram



while the right side is the universal object mapping into the diagram



and these two diagrams depict equivalent data. Namely, one can transform the second diagram into the first without losing any data by removing the upper $\operatorname{Spec}\mathbb{Z}$, reversing the isomorphisms $X \to X$, and contracting the compositions $U_i \to X \to X$ into single arrows $U_i \to X$. Since $X \to Y$ is a closed immersion, so therefore is

$$U_1 \cap U_2 = (U_1 \times_{\operatorname{Spec}\mathbb{Z}} U_2) \times_Y X \to U_1 \times_{\operatorname{Spec}\mathbb{Z}} U_2$$

by the previous problem. But the target of this closed immersion is affine: it is $\operatorname{Spec}(\mathcal{O}(U_1) \otimes_{\mathbb{Z}} \mathcal{O}(U_2))$. By the previous problem again, $U_1 \cap U_2$ is therefore also affine.

(b) Note that the fiber of (2) under $X \to \operatorname{Spec}\mathbb{Z}$ contains 2 points, while the fiber of (2) under $X \times_{\operatorname{Spec}\mathbb{Z}} X \to \operatorname{Spec}\mathbb{Z}$ contains 4 points. All 4 of these points are in the closure of $\Delta(X)$, but only two of them belong to $\Delta(X)$. So Δ is not a homeomorphism of X onto a closed subset of $X \times_{\operatorname{Spec}\mathbb{Z}} X$, and so cannot be a homeomorphism. (Compare Hartshorne example II.4.0.1.)

6. (a) Locally on Y, a closed immersion has the form $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$, $\mathcal{F}$ corresponds to an A/I-module, and $i_*\mathcal{F}$ corresponds to the same module but now viewed over A via the map $A \to A/I$. So it's again quasicoherent. (Or if you prefer, a closed immersion is quasicompact and quasiseparated, so the criterion from an earlier homework applies.)

   (b) Same argument as in (a).

   (c) This follows from problem 4(a).

   (d) If we use coverings as in (c), then we can match up the Čech complexes term-by-term, so the cohomology must also match.

7. (a) As suggested in the hint, we would like to define for each open set U a sheaf $\mathcal{O}_U$ of abelian groups such that the morphisms $\mathcal{O}_U \to \mathcal{F}$ of sheaves correspond to elements of $\mathcal{F}(U)$. In particular, we should have

$$\mathcal{O}_{U,x} = \begin{cases} \mathbb{Z} & x \in U \\ 0 & x \notin U \end{cases}.$$

   One might be tempted to try a direct sum of skyscraper sheaves but that won't work. The right construction is the *extension by zero*, which is the sheafification $\mathcal{O}_U$ of the presheaf defined by

$$V \mapsto \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & V \not\subseteq U. \end{cases}$$

   Now the inclusion $U \subseteq V$ defines an injection $\mathcal{O}_U \to \mathcal{O}_V$, so every morphism $\mathcal{O}_U \to \mathcal{F}$ extends to a morphism $\mathcal{O}_U \to \mathcal{F}$. That indeed means that every element of $\mathcal{F}(U)$ extends to an element of $\mathcal{F}(V)$, so $\mathcal{F}$ is flasque.

   (b) Let $s \in \mathcal{H}(X)$ be a section. Since $\mathcal{G} \to \mathcal{H}$ is surjective, we can find an open cover $\{U_i\}_{i \in I}$ of X such that the restriction of s to U_i lifts to some section $t_i \in \mathcal{G}(U_i)$. Use the axiom of choice to impose a well ordering on the index set I (or assume X is quasicompact so that I is only a finite set), and put $V_i = \cup_{j<i} U_i$. We will build a coherent sequence of sections $u_i \in \mathcal{G}(V_i)$ lifting s as follows.

   - For i the initial element of I, we put $u_i = 0$.
   - For i a limit element, we apply the sheaf axiom to glue the u_j for $j < i$ together to make u_i. (This step doesn't occur in the quasicompact case.)

- For i a successor element, note that $u_{i-1} - t_i \in \mathcal{G}(U_i \cap V_i)$ maps to zero in $\mathcal{H}(U_i \cap V_i)$, so it is the image of an element $v_i \in \mathcal{F}(U_i \cap V_i)$. Since $\mathcal{F}$ is flasque, v_i is the restriction of some $w_i \in \mathcal{F}(U_i)$, which we will also view as an element of $\mathcal{G}(U_i)$. Then w_i maps to zero in $\mathcal{H}(U_i)$, so $t_i + w_i \in \mathcal{G}(U_i)$ also lifts s, but it also agrees with u_{i-1} in $\mathcal{G}(U_i \cap V_i)$. So we can glue $t_i + w_i$ and u_{i-1} together to get u_i.