Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 6 (due Wednesday, February 20)

Solve the following problems, and turn in the solutions to *four* of them. If and only if you've never seen the snake lemma before, please submit problem 1 as one of your four, and try it yourself before checking any references.

Speaking of references, I'd like to remind you that collaboration and use of references is permitted but should be cited. Aside from the issue of academic integrity, I'd like to know what resources you find helpful!

1. Prove the *snake lemma* as follows. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of abelian groups (or modules over a ring) with exact rows.

(a) Prove that there is a well-defined homomorphism $\delta : \ker(f_3) \to \operatorname{coker}(f_1)$ characterized as follows: given $a_3 \in \ker(f_3)$, if we lift it to $a_2 \in A_2$ and map to $b_2 \in B_2$, then b_2 lifts to $b_1 \in B_1$ and $\delta(a_3)$ is the class of b_1. The point is to check that $\delta(a_3)$ does not depend on any choices made along the way (the only real choice was a_2).

(b) Prove that the sequence

$$0 \to \ker(f_1) \to \ker(f_2) \to \ker(f_3) \xrightarrow{\delta} \operatorname{coker}(f_1) \to \operatorname{coker}(f_2) \to \operatorname{coker}(f_3) \to 0$$

is exact, where δ is as in (a) and the other maps are the obvious ones induced by the diagram.

2. Let k be an algebraically closed field. Let $\mathcal{F}$ be a vector bundle of rank 2 over $\mathbb{P}^1_k$.

(a) Suppose that that there exists a short exact sequence

$$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

in which $n_1 \geq n_2$. Prove that the exact sequence splits. Hint: reduce to the case $n_2 = 0$, then use the long exact sequence in cohomology.

(b) Suppose that there exists a short exact sequence

$$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

in which $n_1 \leq n_2 - 1$. Prove that there also exists a short exact sequence

$$0 \to \mathcal{O}(n_1 + c) \to \mathcal{F} \to \mathcal{O}(n_2 - c) \to 0$$

for some positive integer c. Hint: this time, reduce to the case $n_1 = -1$ and remember that every line bundle on $\mathbb{P}^1_k$ of degree n is isomorphic to $\mathcal{O}(n)$. But be careful: the quotient of two vector bundles is not always a vector bundle!

3. Let k be an algebraically closed field. Let $\mathcal{F}$ be a vector bundle of rank d over $\mathbb{P}^1_k$.

   (a) Prove that there exists a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F}$ of $\mathcal{F}$ by vector subbundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to $\mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$. Hint: use the fact that $\mathcal{F}(n)$ is generated by global sections for n sufficiently large.

   (b) Using (a) and the previous exercise, prove that there exists an isomorphism

   $$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d)$$

   for some $n_1, \ldots, n_d \in \mathbb{Z}$ (but not necessarily the same ones you found in (a)). Hint: note that the sum of the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ is independent of the filtration. Then use the previous exercise to raise the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for small values of i at the expense of larger values.

4. Recall that a morphism $f : X \to Y$ of schemes is a *closed immersion* if f is a homeomorphism of X to a closed subset of Y and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

   (a) Recall that for any ring A and any ideal I, the map $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a closed immersion. Prove that conversely, any closed immersion into $\mathrm{Spec}(A)$ has this form. Hint: view the kernel of $f^\sharp$ as a quasicoherent sheaf.

   (b) Prove that the property of being a closed immersion is stable under base change. That is, if $Y \to X, Z \to X$ are two morphisms of schemes and $Y \to X$ is a closed immersion, then $Y \times_X Z \to Z$ is a closed immersion. Hint: reduce to the case where both X and Z are affine. (We'll see many more statements like this soon.)

5. Recall that a scheme X is *separated* if the diagonal map $\Delta : X \to X \times_{\mathrm{Spec}\,\mathbb{Z}} X$ is a closed immersion.

   (a) Let X be a separated scheme. Prove that for any two open affine subschemes U_1, U_2 of X, the intersection $U_1 \cap U_2$ is again affine. Hint: write the intersection as a fibred product.

   (b) Let X be the scheme obtained by glueing two copies of $\mathrm{Spec}\,\mathbb{Z}$ together along $\mathrm{Spec}\,\mathbb{Z}[1/2]$. Prove that X is not separated. This shows that (a) does not have a converse.

6. Let $i : X \to Y$ be a closed immersion. Let $\mathcal{F}$ be a quasicoherent sheaf on X.

   (a) Prove that $i_*\mathcal{F}$ is a quasicoherent sheaf on Y.

   (b) Suppose that $\mathcal{F}$ is locally finitely generated. Prove that $i_*\mathcal{F}$ is also finitely generated.

   (c) Let $\{U_i\}_{i \in I}$ be a covering of Y by opens such that every finite intersection is affine. Prove that $\{X \times_Y U_i\}_{i \in I}$ is a covering of X such that every finite intersection is affine.

(d) Prove that $H^i(X, \mathcal{F}) = H^i(Y, i_*\mathcal{F})$ for all i, where the cohomology is computed as Čech cohomology using coverings of X and Y as in (c).

7. A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for any open subsets $U \subseteq V$, the restriction map from $\mathcal{F}(V)$ to $\mathcal{F}(U)$ is surjective.

   (a) Let $\mathcal{F}$ be an injective sheaf of abelian groups. Prove that $\mathcal{F}$ is also flasque. Hint: find a sheaf $\mathcal{O}_U$ of abelian groups for which

   $$\mathcal{O}_{U,x} = \begin{cases} \mathbb{Z} & x \in U \\ 0 & x \notin U \end{cases}$$

   and for which morphisms $\mathcal{O}_U \to \mathcal{F}$ correspond to elements of $\mathcal{F}(U)$.

   (b) Prove that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves of abelian groups on X and $\mathcal{F}$ is flasque, then

   $$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

   is exact. If you get stuck, you may assume that X is quasicompact (the general case uses some axiom-of-choice magic).

   (c) Optional (don't turn this in): assume X is a locally ringed space and reprove (a) for sheaves of $\mathcal{O}_X$-modules. If you're on a roll, look up a proof that the category of modules over a ring has enough injectives, then prove that the category of $\mathcal{O}_X$-modules does too.