Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 7 (due Wednesday, February 27)

Solve the following problems, and turn in the solutions to *four* of them, including *no more than two* of problems 5–7. (Surgeon general's warning: too much abstraction may be harmful to your mathematical health!)

Reminder: to solve a given problem, you may (and will often need to) cite any preceding problem on this homework or any problem on any prior homework, whether or not you turned in the prior problem. However, on this problem set only, please do not cite any results from the handout "Notes on sheaf cohomology and Čech cohomology", because the intended logical dependence is the reverse (the arguments there depend on results from this homework). On the other hand, you are free to emulate arguments you find in that handout!

Also, in addition to inanimate references, please remember to indicate any collaboration with classmates or anyone else (except me) you may have asked about the problems.

1. A morphism $f : X \to Y$ is *finite* if locally on Y, it has the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where B is a finite A-algebra (or more precisely a "module-finite" A-algebra). Prove that if f is finite and Y is affine, then X is affine and $\mathcal{O}(X)$ is a finite $\mathcal{O}(Y)$-algebra; that is, any finite map to $\mathrm{Spec}(A)$ is isomorphic to $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for some (module-)finite A-algebra. Hint: look at $f_*\mathcal{O}_X$ as a $\mathcal{O}_Y$-module.

2. Let k be an algebraically closed field of characteristic other than 2. Let $P(x)$ be a monic polynomial of degree $2g+1$ over k with no repeated roots. Let X be the curve $y^2 = P(x)$ in $\mathbb{A}^2_k$. Let Y be the smooth projective curve containing X, so that $Y - X$ consists of a single point ∞.

   (a) Put $s = dx/(2y)$. Prove that $x^i s \in H^0(Y, \omega_Y)$ for $i = 0, \dots, g-1$. Hint: compute the orders of x and y at ∞ using the fact that any principal divisor has degree 0.

   (b) Prove that the cokernel of $d : H^0(X, \mathcal{O}_Y) \to H^0(X, \omega_Y)$ (the "first de Rham cohomology group of X") is generated by $x^i s$ for $i = 0, \dots, 2g-1$. Hint: use the relations $d(x^n y) = 0$ to reduce degrees. (It turns out that one can use (a) and duality to show that these elements form a basis of the cokernel, but you don't have to do this.)

3. Let R be a ring and put $X = \mathbb{P}^n_R$. Recall that there exists an exact sequence

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0$$

(see for instance Lemma 7.4.15 of Gathmann). Prove that for $j = 1, \dots, n$, there also exists an exact sequence

$$0 \to \wedge^j \Omega_X \to \mathcal{O}_X(-j)^{\oplus\binom{n+1}{j}} \to \wedge^{j-1}\Omega_X \to 0.$$

Hint: you might find it helpful to think about what happens when you start with a short exact sequence

$$0 \to M \to N \to R \to 0$$

of free modules over R and take exterior powers.

4. Let R be a ring and put $X = \mathbb{P}^n_R$. Using the previous exercise, compute $H^i(\mathbb{P}^n_R, \Omega^j)$ for all i, j, n. You should get $H^i(\mathbb{P}^n_R, \Omega^j) = 0$ unless $i = j$.

5. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X.

   (a) Prove that there exists an exact sequence

   $$0 \to \mathcal{F} \to \mathcal{I}_0 \xrightarrow{d_0} \mathcal{I}_1 \xrightarrow{d_1} \cdots$$

   in which the $\mathcal{I}_i$ are injective (an *injective resolution* of $\mathcal{F}$). Hint: use a problem from a previous homework.

   (b) Let

   $$0 \to \mathcal{G} \to \mathcal{J}_0 \xrightarrow{e_0} \mathcal{J}_1 \xrightarrow{e_1} \cdots$$

   be another exact sequence of sheaves (but the $\mathcal{J}_i$ need not be injective). Prove that any morphism $f : \mathcal{G} \to \mathcal{F}$ fits into a a commutative diagram

   $$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "f"] & \mathcal{J}_0 \arrow[r, "e_0"] \arrow[d, "f_0"] & \mathcal{J}_1 \arrow[r, "e_1"] \arrow[d, "f_1"] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}_0 \arrow[r, "d_0"] & \mathcal{I}_1 \arrow[r, "d_1"] & \cdots \end{tikzcd}$$

   (c) Set notation as in (b) and suppose that $f = 0$. Prove that there exist maps $h_i : \mathcal{J}_i \to \mathcal{I}_{i-1}$ for $i > 0$ such that

   $$\begin{aligned} f_0 &= h_1 \circ e_0 \\ f_i &= h_{i+1} \circ e_i + d_{i-1} \circ h_i \qquad (i > 0). \end{aligned}$$

   This is an example of a *chain homotopy.*

6. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Choose an injective resolution $\mathcal{I}.$ of $\mathcal{F}$ and define the *sheaf cohomology* groups

   $$H^i(X, \mathcal{F}) = \frac{\ker(\mathcal{I}_i(X) \to \mathcal{I}_{i+1}(X))}{\operatorname{im}(\mathcal{I}_{i-1}(X) \to \mathcal{I}_i(X))}.$$

   (a) Prove that this definition is functorial for a fixed choice of resolution. That is, given a morphism $f : \mathcal{G} \to \mathcal{F}$, fix an injective resolution of $\mathcal{G}$, and show that if we use part (b) of the previous exercise to define maps $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F})$, then these maps only depend on f. Hint: reduce to the case $f = 0$.

   (b) Use (a) to show that that the definition of $H^i(X, \mathcal{F})$ is independent of the choice of the resolution; that is, given two definitions made using two different resolutions, there is a distinguished isomorphism between them. (It is possible to do this using universal δ-functors, but I said not to use results from the sheaf cohomology handout!)

(c) Prove that if $\mathcal{F}$ is injective, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

7. Let $\mathcal{F}$ be a flasque sheaf (not necessarily injective) on a topological space X.

   (a) Prove that $\check{H}^1(X, \mathcal{F}, \underline{U}) = 0$ for any open covering $\underline{U} = \{U_i\}_{i \in I}$. You may assume X is quasicompact if necessary. Hint: this is similar to a previous homework problem.

   (b) Prove that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Hint: embed $\mathcal{F}$ into an injective sheaf and induct on i, being careful about the base case $i = 1$.