Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 8

1. (a) Given a morphism $X \to \operatorname{Spec}(R)$ of locally ringed spaces, pullback of global sections defines a map $R \to \mathcal{O}(X)$. In the other direction, given a homomorphism $R \to \mathcal{O}(X)$, we define a map $h : X \to \operatorname{Spec}(R)$ of sets by taking x to the inverse image of $\mathfrak{m}_{X,x}$ under $R \to \mathcal{O}(X) \to \mathcal{O}_{X,x}$. To check that this map is continuous, it suffices to check that the inverse image of any distinguished open set $\operatorname{Spec}(R_f)$ is open (since these form a basis for the topology of $\operatorname{Spec}(R)$). This inverse image consists of the points $x \in X$ for which f maps to a unit in $\mathcal{O}_{X,x}$. This set X_f is open: if f maps to a unit u in $\mathcal{O}_{X,x}$, then the inverse v can be found over some neighborhood of x, and the difference $uv - 1$ equals zero over some smaller neighborhood of x.

   To define a morphism of ringed spaces, we must define a morphism $\mathcal{O}_{\operatorname{Spec}(R)} \to h_*\mathcal{O}_X$ of sheaves of rings. For $f \in R$, note that the image of f in $\mathcal{O}(X)$ is invertible in $\mathcal{O}(X_f)$: in a neighborhood of each point of X_f there is an inverse of f, and these inverses must glue. We thus get maps

$$R_f = \mathcal{O}_{\operatorname{Spec}(R)}(D(f)) \to \mathcal{O}(X_f) = h_*\mathcal{O}_X(D(f))$$

   and by sheafifying we get the map on sheaves.

   Finally, note that this is indeed a map of locally ringed spaces: namely, this is clear from the pointwise definition. If $x \in X$ maps to $\mathfrak{p} \in \operatorname{Spec}(R)$, then by definition $\mathfrak{p}$ is the inverse image of $\mathfrak{m}_{X,x}$ under $R \to \mathcal{O}(X) \to \mathcal{O}_{X,x}$, so $\mathfrak{p}R_{\mathfrak{p}}$ maps into $\mathfrak{m}_{X,x}$ and the homomorphism $R_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is indeed a local homomorphism of local rings.

   (b) It is clear that we have a ringed space. To check that it is locally ringed, note that for each $x \in X$, if a holomorphic function f on some neighborhood of x is nonzero at x, then it is nonzero in a neighborhood of x and on any such neighborhood f^{-1} is holomorphic. Therefore the kernel of evaluation at x is indeed a maximal ideal, so $\mathcal{O}_{X,x}$ is a local ring.

   (c) For $i = 0, \dots, n$, the subset $\{[x_0 : \cdots : x_n] \in X : x_i \neq 0\}$ maps to the distinguished open affine subspace $D(x_i)$ of $\mathbb{P}^n_{\mathbb{C}}$ using (a), because polynomial functions on $\mathbb{C}^n$ are holomorphic.

2. (a) If we write the matrices in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   and identify Z with $\operatorname{Spec} k[a,b,c,d]$, then $X = \operatorname{Spec} k[a,b,c,d,e]/(e(ad-bc)-1)$ because a matrix is invertible if and only if its determinant is invertible.

(b) The action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1_k$ is by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [t_0 : t_1] = [at_0 + bt_1 : ct_0 + dt_1].$$

So we get a surjection onto $\mathbb{P}^1_k$ by looking at the orbit of $[1 : 0]$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [a : c].$$

(c) We have shown previously that $H^0(\mathbb{P}^1_k, \mathcal{O}) = k$, so if $\mathbb{P}^1_k$ were affine it would be isomorphic to $\mathrm{Spec}(k)$. But of course it isn't because it contains more than one point!

3. Let $t_0, \dots, t_n$ be the homogeneous coordinates on $\mathbb{P}^n_{\mathbb{Z}}$, and let $t'_0, \dots, t_n$ and $t''_0, \dots, t''_n$ be the homogeneous coordinates on the two factors of $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$. Then $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ is covered by the open affines

$$\begin{aligned} D(t'_i, t''_j) &= D(t'_i) \times_{\mathrm{Spec}\,\mathbb{Z}} D(t'_j) \\ &= \mathrm{Spec}\, k[t'_k/t'_i, t''_l/t''_j : k \neq i, j \neq l]. \end{aligned}$$

The image of the diagonal is the closed subspace cut out by $(t'_j/t'_i)(t''_i/t''_j) - 1$ and $t'_k/t'_i - (t''_k/t''_j)(t'_j/t'_i)$ for all k.

4. (a) Property (i): if $f^{-1}(\mathrm{Spec}(R))$ is covered by $\mathrm{Spec}(S_1), \dots, \mathrm{Spec}(S_n)$, then $f^{-1}(\mathrm{Spec}(R_g))$ is covered by $\mathrm{Spec}(S_{1,g}), \dots, \mathrm{Spec}(S_{n,g})$. Property (ii): if $f^{-1}(\mathrm{Spec}(R_{g_i}))$ is covered by finitely many affines, then they all together cover $f^{-1}(\mathrm{Spec}(R))$.

(b) Property (i): If $f^{-1}(\mathrm{Spec}(R)) = \mathrm{Spec}(S)$, then $f^{-1}(\mathrm{Spec}(R_g)) = \mathrm{Spec}(S_g)$. Property (ii): if $f^{-1}(\mathrm{Spec}(R_{g_i})$ is affine, then these sets satisfy the criterion from PS 2 problem 1, so $f^{-1}(\mathrm{Spec}(R))$ is affine.

(c) Property (i): If $f^{-1}(\mathrm{Spec}(R))$ is covered by $\mathrm{Spec}(S_i)$ with S_i of finite type, then $f^{-1}(\mathrm{Spec}(R_g))$ is covered by $\mathrm{Spec}(S_{i,g})$. Property (ii): if $f^{-1}(\mathrm{Spec}(R_{g_i})$ is covered by some $\mathrm{Spec}(S_{i,j})$ with $S_{i,j}$ a finitely generated ring over R_{g_i}, then $S_{i,j}$ is also a finitely generated ring over R because $R_{g_i} = R[x]/(xg_i - 1)$ is finitely generated! So just take these together to cover $f^{-1}(\mathrm{Spec}(R))$.

(d) By (c), we may assume $X = \mathrm{Spec}(R)$ is affine. Property (i): if S is a finitely generated R-algebra, then so is S_g. Property (ii): if S_{g_i} are finitely generated R_{g_i}-algebras, then by clearing denominators we can choose $s_{i,j} \in S$ which generate S_{g_i} over R. Let S' be the R-subalgebra of S generated by all of the $s_{i,j}$; then the map $S' \to S$ of R-modules is surjective on stalks and so must be an isomorphism.

(e) The contradiction is that the infinite disjoint union of $\mathrm{Spec}(k)$ does map to Spec of the infinite direct sum by problem 1(a), but the map is not an isomorphism. In the case $k = \mathbb{F}_2$, this is related to the existence of *ultrafilters*.

5. (a) Because of the uniqueness, it is enough to check the case when $X = \mathrm{Spec}(R)$ is affine. In this case, $Y = V(I)$ for some ideal I of R, and we can and must take $Z = \mathrm{Spec}(R/\sqrt{I})$.

   (b) Take $Y = X$ in (a).

   (c) Because of the uniqueness, we may assume both $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are affine. Any nilpotent element of R must map to a nilpotent element of S, which is zero because Y is reduced. Hence $R \to S$ factors uniquely through $R/\sqrt{(0)} \to S$, as claimed.

   (d) To produce g_{red}, apply (c) to factor $g \circ f_Y : Y_{\mathrm{red}} \to X$ through f_X. For functoriality, compare the squares corresponding to $Y \to X$, $Z \to Y$, and $Z \to X$.

6. Finite implies affine (by a previous homework) and finite type (obvious). Affine implies separated (proved in class). So we need only check universally closed. Since finite is stable under base change, we need only check that finite implies closed; that is, we must check that if S is a finite R-algebra then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a closed map. By the previous exercise, we may assume R and S are reduced. Since affine implies quasicompact, by Lemma 4 from "Projective and proper morphisms" it is enough to check that the image of $\mathrm{Spec}(S)$ is stable under specialization. So pick $x, y \in \mathrm{Spec}(R)$ with y a specialization of x and x in the image of $\mathrm{Spec}(S)$. To check that y is in the image of $\mathrm{Spec}(S)$, we may replace $\mathrm{Spec}(R)$ with the closure of x with the reduced subscheme structure; now $\mathrm{Spec}(R)$ is reduced and irreducible, hence a domain. But now the going-up theorem implies that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.

7. (a) We showed in class that separatedness is stable under base extension. It is obvious that finite type is stable under the base extension, because the property "S is a finitely generated R-algebra" is stable under base extension on R. And universal closedness is by definition stable under base extension.

   (b) First recall that the composition of two closed maps $X \to Y, Y \to Z$ of topological spaces is again a closed map: the image in Z of a closed subset of X is also the image of the image in Y.

   Let $X \to Y$ and $Y \to Z$ be separated morphisms. To check that $X \to Z$ is separated, note that $X \to X \times_Z X$ factors as $X \to X \times_Y X$, which is a closed immersion, followed by $X \times_Y X = (X \times_Z X) \times_{Y \times_Z Y} Y \to X \times_Z X$ which is the base extension of the closed immersion $Y \to Y \times_Z Y$. So the diagonal is the composition of two closed maps, so it's closed. Hence $X \to Z$ is separated.

   Now suppose $X \to Z$ is proper. To check that $X \to Z$ is of finite type, just recall that the property "S is a finitely generated R-algebra" is transitive. To check that $X \to Z$ is universally closed, let $U \to Z$ be any morphism. Then $Y \times_Z U \to U$ and $X \times_Z U = X \times_Y (Y \times_Z U) \to Y \times_Z U$ are base extensions of proper morphisms, so both are closed, and the composition $X \times_Z U \to U$ is thus a closed map.

(c) If $X \to Z$ and $Y \to Z$ are both separated/proper, then $X \times_Z Y \to Z$ is the composition of two separated/proper morphisms $X \times_Z Y \to Y$ (separated/proper by (b)) and $Y \to Z$. By (c), this composition is separated/proper.

(d) Since $Y \to Z$ is separated, $Y \to Y \times_Z Y$ is a closed immersion, hence separated/proper. By identifying $X = (X \times_Z Y) \times_{Y \times_Z Y} Y$, we see that $X \to X \times_Z Y$ is separated/proper. Now $X \times_Z Y \to Y$ is the base extension of the separated/proper morphism $X \to Z$ and so is separated/proper, so the composition $X \to Y$ is separated/proper.

8. Put $Y = \mathrm{Spec}(S)$. Since f is proper, Y is a finitely generated k-algebra. Choose a surjection $k[x_1, \ldots, x_n] \to S$; this defines a closed immersion $Y \to \mathbb{A}^n_k$ over $\mathrm{Spec}(k)$. Follow this with the projection $\mathbb{A}^n_k \to \mathbb{A}^1_k$ defined by x_i, then embed $\mathbb{A}^1_k$ into $\mathbb{P}^1_k$. Since $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ is separated and $Y \to \mathrm{Spec}(k)$ is proper, by the previous problem $Y \to \mathbb{P}^1_k$ is proper, so its image is closed. But this image cannot be all of $\mathbb{P}^1_k$ because by definition Y factors through $\mathbb{A}^1_k$, so it must be a finite set. Since this is true for all i, the closed immersion $Y \to \mathbb{A}^n_k$ must have finite image.

   We may thus reduce to the case where Y consists of a single point. If Y is reduced, it must then be a field which is a finitely generated k-algebra, but by the Nullstellensatz such a field is itself a finite extension of k. In the general case, if I is the nilradical of S, then because S is noetherian, some power I^m of I must be the zero ideal, and S/I, I/I^2, ..., I^{m-1}/I^m are all finite-dimensional over S/I and hence over k. So S has finite length as a k-module, finishing the proof.