Math 203B (Algebraic Geometry), UCSD, winter 2013
Problem Set 8 (due Wednesday, March 6)

Solve the following problems, and turn in the solutions to *four* of them.

1. (a) Let X be an arbitrary locally ringed space (not necessarily a scheme) and let R be a ring. Prove that ring homomorphisms $R \to \mathcal{O}(X)$ are in bijection with morphisms $X \to \operatorname{Spec}(R)$ of locally ringed spaces. More precisely, for $R \to \mathcal{O}(X)$ a ring homomorphism, your corresponding map $X \to \operatorname{Spec}(R)$ should take x to the inverse image of $\mathfrak{m}_{X,x}$ under $R \to \mathcal{O}(X) \to \mathcal{O}_{X,x}$.

   (b) Let X be the topological space $\mathbb{C}^n$ equipped with the sheaf $\mathcal{O}_X$ of holomorphic functions. Check that $\mathcal{O}_{X,x}$ is a locally ringed space. (All you need to know about holomorphic functions is that on any product of open discs, they are the functions defined by convergent Taylor series.)

   (c) Let X be the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation identifying v with cv for all $c \in \mathbb{C} - \{0\}$. Equip X with a sheaf of rings in such a way that for $i = 0, \dots, n$, the subset $\{[x_0 : \cdots : x_n] \in X : x_i \neq 0\}$ is isomorphic to the locally ringed space $\mathbb{C}^n$ of (b) via the map

$$[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right).$$

   Use (a) to produce an injective map $X \to \mathbb{P}^n_{\mathbb{C}}$ of locally ringed spaces. This is an example of the process of *analytification* relating complex algebraic and analytic varieties. We'll use this again on the next homework.

2. In this problem, we show that if $f : X \to Y$ is a morphism of schemes, it is not always true that the image of an open affine subscheme of X is contained in an open affine subscheme of Y. Let k be an algebraically closed field.

   (a) Let Z be the affine 4-space over k identified with the space of 2×2 matrices. Prove that there is an open affine subscheme X of Z whose closed points are the invertible 2×2 matrices over k.

   (b) Construct a surjective morphism $X \to \mathbb{P}^1_k$. Hint: $\mathrm{GL}_2(k)$ acts on $\mathbb{P}^1_k$ via linear fractional transformations.

   (c) Prove that $\mathbb{P}^1_k$ is not affine. Yes, this is "obvious" but you must still provide a reason!

3. Describe the image of the diagonal morphism $\Delta : \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \mathbb{P}^n_{\mathbb{Z}}$ explicitly enough to see that it is closed. By a result stated in class, this implies that $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ is separated.

4. Using the affine communication lemma, prove that the following properties of a morphism $f : Y \to X$ of schemes are local on the base.

(a) Quasicompact: the inverse image of any open affine subspace of X is quasicompact.

(b) Affine: the inverse image of any open affine subspace of X is an open affine subspace of Y. Hint: check your older homework!

(c) Locally of finite type: for every open affine subscheme $\mathrm{Spec}(R)$ of X, every open affine subscheme of $f^{-1}(\mathrm{Spec}(R))$ is $\mathrm{Spec}(S)$ for S a finitely generated R-algebra.

(d) Locally of finite type, but now check that it is also local on the source.

(e) Optional: find the flaw in this reasoning. For k a field, an infinite disjoint union of copies of $\mathrm{Spec}(k)$ is locally of finite type, but an infinite direct sum of copies of k is not finitely generated as an k-algebra, contradiction.

5. Prove Lemma 1 on the handout "Projective and proper morphisms".

6. Prove that any finite morphism is proper. Hint: use arguments from the handout "Projective and proper morphisms".

7. Throughout this problem, when I write "separated/proper", that means you should check the statement once reading "separated" everywhere, and once more reading "proper" everywhere.

   (a) Prove that properness is stable under base extension. (We showed in class that separatedness is stable under base extension.)

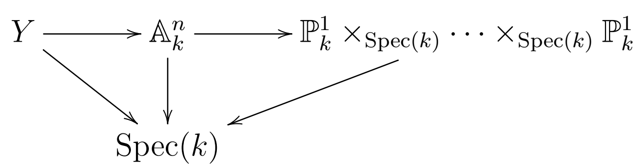
   (b) Prove that a composition of separated/proper morphisms is separated/proper.

   (c) Prove that a product of separated/proper morphisms is separated/proper.

   (d) Suppose $X \to Y \to Z$ are morphisms such that $X \to Z$ is separated/proper and $Y \to Z$ is separated. Prove that $X \to Y$ is also separated/proper.

   Hint: parts (c) and (d) are formal. For (d), notice that the graph morphism $X \to X \times_Z Y$ is a base extension of the diagonal $Y \to Y \times_Z Y$.

8. Suppose that $X = \mathrm{Spec}(k)$ for k a field. Let $f : Y \to X$ be a proper morphism such that Y is affine. Prove that f is finite. Hint: build a diagram



   project Y onto the individual factors of $\mathbb{P}^1_k$, and use a previous exercise. (The finiteness is also true for general X, but you don't have to prove this.)