Math 203B (Algebraic Geometry), UCSD, winter 2013
Solutions for problem set 9

1. (a) Let k be a field and let U be the complement of the origin in $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. Then
$$\mathcal{O}(U) = k[x,y,y^{-1}] \cap k[x,x^{-1},y] = k[x,y]$$
but the induced map $U \to \operatorname{Spec} k[x,y]$ is not an isomorphism, so U cannot be affine. We thus obtain an example of the desired form by glueing two copies of $\mathbb{A}^2_k$ together along U.

   (b) For any point $x \in X \times_{\operatorname{Spec}\mathbb{Z}} X$, let y,z be the images of x under the two projections to X. By hypothesis, y and z are contained in some open separated subscheme U of X, so $x \in U \times_{\operatorname{Spec}\mathbb{Z}} U$. Therefore $X \times_{\operatorname{Spec}\mathbb{Z}} X$ is covered by the open subschemes $U \times_{\operatorname{Spec}\mathbb{Z}} U$, so to check that the image of $\Delta : X \times_{\operatorname{Spec}\mathbb{Z}} X$ is closed it suffices to check that its intersection with each $U \times_{\operatorname{Spec}\mathbb{Z}} U$ is closed. But that intersection is just the image of $\Delta_U : U \to U \times_{\operatorname{Spec}\mathbb{Z}} U$, which is closed because U is separated.

   (c) To apply (b), it suffices to check that for any two points x,y of $\mathbb{P}^n_{\mathbb{Z}}$, we can find a homogeneous polynomial f whose zero locus contains neither point, as then the distinguished open subset $D(f)$ will be affine and hence separated. The easiest way to do this is to let $t_0,\dots,t_n$ be homogeneous coordinates of $\mathbb{P}^n_{\mathbb{Z}}$ and choose indices i,j for which t_i does not vanish at x (i.e., does not belong to the maximal ideal of the local ring of $\mathbb{P}^n_{\mathbb{Z}}$ at x) and t_j does not vanish at y; then one of
$$t_i, t_j, t_i + t_j$$
must work. (If neither t_i nor t_j works, then t_i must vanish at y so $t_i + t_j$ does not, and similarly t_j must vanish at x so $t_i + t_j$ does not.)

2. (a) It is clear that (iv) implies (i) and that (iii) implies (ii); in addition, (i) implies (iii) because of the definition of $\operatorname{Ext}^i_R(M,N)$ as a derived functor for $\operatorname{Hom}_R(\bullet, N)$. It thus remains to check that (ii) implies (iv).

   Assuming (iv), given a projective resolution of M, for any R-module N, we have
$$0 = \operatorname{Ext}^{m+1}_R(M,N) = \frac{\ker(\operatorname{Hom}_R(P_{m+1},N) \to \operatorname{Hom}_R(P_{m+2},N))}{\operatorname{im}(\operatorname{Hom}_R(P_m,N) \to \operatorname{Hom}_R(P_{m+1},N))}.$$
In particular, for
$$N = \frac{P_{m+1}}{\operatorname{im}(P_{m+2} \to P_{m+1})} = \frac{P_{m+1}}{\ker(P_{m+1} \to P_m)},$$
the obvious projection $P_{m+1} \to N$ lifts to a map $P_m \to N$ splitting the inclusion $N \to P_m$. Consequently, P_m splits as a direct sum of N and
$$\frac{P_m}{N} \cong \frac{P_m}{\operatorname{im}(P_{m+1} \to P_m)},$$
so the latter is a projective R-module. This yields (ii).

(b) Consider the projective resolution

$$\cdots \to R \to R \to R \to M \to 0$$

in which the maps $R \to R$ are all multiplication by x. At each step, the quotient $R/\operatorname{im}(R \to R)$ is $R/xR \cong M$, which is not projective because $R \to M$ is surjective but $\operatorname{Hom}_R(M, R) \to \operatorname{Hom}_R(R, R) \cong R$ is not (its image is only xR). Thus condition (iv) fails for all m.

3. (a) The statement says that any finite module over $R = k[x_1, \ldots, x_n]$ has projective dimension $\leq n$.

   (b) By results from class, there exists an exact sequence

$$\mathcal{G}_{n+1} \to \cdots \to \mathcal{G}_1 \to \mathcal{F} \to 0$$

in which each $\mathcal{G}_i$ is a direct sum of various $\mathcal{O}(d)$, and in particular is a vector bundle. Over each distinguished open subset $D(x_i)$, we have an equality

$$\operatorname{coker}(\mathcal{G}_{n+1} \to \mathcal{G}_n))(D(x_i)) = \operatorname{coker}(\mathcal{G}_{n+1}(D(x_i)) \to \mathcal{G}_n(D(x_i)))$$

because $D(x_i)$ is an affine scheme. But since $D(x_i) = \operatorname{Spec} k[x_j/x_i : j \neq i]$, by (a) the module $\mathcal{F}(D(x_i))$ is of projective dimension $\leq n$, so the cokernel appearing above is a projective module over $\mathcal{O}(D(x_i))$. We may thus replace $\mathcal{G}_n$ by the cokernel of $\mathcal{G}_{n+1} \to \mathcal{G}_n$ to get the desired exact sequence.

4. Put $R_0 = \mathbb{C}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$; then both R_0 and R are noetherian local rings with the same completion $S = \mathbb{C}[\![x_1, \ldots, x_n]\!]$. But the completion of a noetherian local ring is faithfully flat (see Chapter 10 of Atiyah-Macdonald), so S is faithfully flat over both R_0 and R. But this implies that R is faithfully flat over R_0.

5. (a) We proceed by induction on i. Assume either that $i = 0$ or that $i > 0$ and the claim is known for $i - 1$. Let H be a hyperplane; we then have a short exact sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}_H(d-1) \to 0.$$

By comparing long exact sequences and using the induction hypothesis, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(d-1)) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(d)) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}_H(d-1)) \arrow[r] \arrow[d] & H^{i+1}(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(d-1)) \\
0 \arrow[r] & H^i(X, j^*\mathcal{F}(d-1)) \arrow[r] & H^i(X, j^*\mathcal{F}(d)) \arrow[r] & H^i(X, \mathcal{F}_H(d-1)) &
\end{tikzcd}$$

with exact rows (note that the case $i = 1$ is special but still works). In this diagram, the third vertical arrow is an isomorphism. If the second vertical arrow is an isomorphism, so is the first; since the isomorphism is known for $d = 0$ by assumption, it follows for all $d \leq 0$. On the other hand, if $\mathcal{F} = \mathcal{O}$, $d > 0$, and the first vertical arrow is an isomorphism, then $H^{i+1}(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d-1)) = 0$ and so the second vertical arrow is an isomorphism; so if $\mathcal{F} = \mathcal{O}$ then the isomorphism also follows for all $d > 0$.

(b) We proceed by induction on i. Assume either that $i = 0$ or that $i > 0$ and the claim is known for $i - 1$. Since $\mathcal{F}$ is locally free, we can make an exact sequence

$$0 \to \mathcal{F} \to \bigoplus_h \mathcal{O}(d_h) \to \mathcal{G} \to 0$$

for some $d_h \in \mathbb{Z}$ by dualizing the usual construction, and $\mathcal{G}$ will also be locally free. For d large, we may twist and take the long exact sequence to get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(d)) \arrow[r] \arrow[d] & \bigoplus_h H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d+d_h)) \arrow[r] \arrow[d] & H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{G}(d)) \arrow[d] \\ 0 \arrow[r] & H^i(X, j^*\mathcal{F}(d)) \arrow[r] & \bigoplus_h H^i(X, j^*\mathcal{O}(d+d_h)) \arrow[r] & H^i(X, j^*\mathcal{G}(d)) \end{tikzcd}$$

with exact rows (note that the case $i = 1$ is special but still works). Of the first three vertical arrows, the second is an isomorphism by (i) and (ii) and (a), so the first is injective, as then is the third by repeating the argument with $\mathcal{G}$ in place of $\mathcal{F}$. By diagram chasing, the first vertical arrow is also surjective, as then is the third by repeating the argument with $\mathcal{G}$ in place of $\mathcal{F}$. Thus the isomorphism holds for $\mathcal{F}(d)$ for d large, and hence for $\mathcal{F}$ by (a).

(c) By a previous exercise, there exists an exact sequence

$$0 \to \mathcal{G}_m \to \cdots \to \mathcal{G}_0 \to \mathcal{F} \to 0$$

in which the $\mathcal{G}_m$ are vector bundles. We induct on m. If $m = 0$, then $\mathcal{F}$ is a vector bundle and (b) applies. Otherwise, take cohomology of $0 \to \mathcal{H} \to \mathcal{G}_0 \to \mathcal{F} \to 0$ and apply the induction hypothesis to $\mathcal{H}$ and apply (b) to $\mathcal{G}_0$. (Here we need (iii) to be sure that j^* preserves the exact sequence.)

6. Strictly speaking, one should do induction over the whole argument; that is, to prove (a) we assume (c) in dimension one lower.

(a) Let H be a hyperplane through x. By the induction hypothesis, $\mathcal{F}_H(d)_x$ is generated by finitely many global sections for sufficiently large d. By Nakayama's lemma and the overall induction hypothesis, these lift to generators of $\mathcal{F}(d)_x$.

(b) By (a), each point $x \in X$ has a neighborhood on which $\mathcal{F}(d)$ is generated by finitely many global sections for some large d (and hence for all sufficiently large d). Since X is compact, we may cover X with finitely many such neighborhoods and pool the generators to get finitely many generators of all of $\mathcal{F}$.

(c) By (b), we may write $\mathcal{F}$ globally as the cokernel of a map $\mathcal{G} \to \mathcal{H}$ between finite free $\mathcal{O}_X$-modules. We may view $\mathcal{G}$ and $\mathcal{H}$ as the pullbacks of finite free modules $\mathcal{G}_0, \mathcal{H}_0$ over $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$; by the previous problem applied to the Hom sheaf, the map itself also comes from $\mathbb{P}^n_{\mathbb{C}}$. We may then form the cokernel $\mathcal{F}_0$, whose pullback coincides with $\mathcal{F}$ (using faithful flatness of the local rings from problem 4).

7. Let H be an analytic hypersurface in X. Then the ideal sheaf on X defining H is locally the cokernel of a map between finite free $\mathcal{O}_X$-modules, so by the second GAGA exercise it is in fact the pullback of a sheaf on $\mathbb{P}^n_{\mathbb{C}}$. That sheaf in turn defines an algebraic hypersurface whose analytification is H.