Math 203b: Algebraic Geometry (UC San Diego, winter 2013)
Kiran S. Kedlaya
Notes on the Riemann-Roch theorem

In these notes, we explain to how use sheaf cohomology to prove the Riemann-Roch theorem. Note that we need to be careful not to use Riemann-Roch implicitly in the arguments! (I believe this proof is due to Weil; my exposition is plagiarized from some notes of Ravi Vakil.)

First, some notation. Let k be an algebraically closed field. Let X be a smooth projective curve over k with function field K. The letter D will always denote a divisor on X. Let ω_X be the canonical sheaf. Let $g = \dim_k H^0(X, \omega_X)$ be the genus of X, also called the *geometric genus* in order to distinguish it from $g' = \dim_k H^1(X, \mathcal{O}_X)$. The latter is called the *arithmetic genus*; at the end the two genera will turn out to be equal, but we cannot use this fact in the proof!

Lemma 1. *The k-vector spaces $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$ have finite dimension.*

Proof. Proved in lecture. See also Hartshorne, Theorem III.5.2. □

We will have two different types of *Euler characteristic* which will ultimately coincide, but again we will not discover this until later. Define

$$\begin{aligned}\chi_a(D) &= \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D))\\ \chi_g(D) &= \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \omega_X(-D)).\end{aligned}$$

Using χ_a, we can already derive the Riemann inequality except with g replaced by g'.

Lemma 2. *For all D,*

$$\chi_a(D) = \deg(D) + 1 - g'.$$

In particular, $\dim_k H^0(X, \mathcal{O}_X(D)) \geq \deg(D) + 1 - g'$.

Proof. Since the equality is obvious if $D = 0$, it is enough to check that for any closed point $P \in X$,

$$\chi_a(D + P) - \chi_a(D) = 1.$$

To see this, recall that from the short exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + P) \to k_P \to 0 \tag{1}$$

we get a long exact sequence

$$\begin{aligned}0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(D + P)) \to H^0(X, k_P) \to\\ \to H^1(X, \mathcal{O}_X(D)) \to H^1(X, \mathcal{O}_X(D + P)) \to H^1(X, k_P) \to 0.\end{aligned}$$

Since $\dim_k H^0(X, k_P) = 1$ and $\dim_k H^1(X, k_P) = 0$, this gives the desired equality. □

For $P \in X$ a closed point and s a meromorphic differential on X, the *residue* of P at s, denoted $\mathrm{Res}_P(s)$, is computed by choosing a uniformizer t at P, writing s as a formal Laurent series $\sum_{i=-m}^{\infty} a_i t^i \, dt$ and extracting the coefficient of $t^{-1}\, dt$. It was shown on a previous homework that this does not depend on the choice of t.

Lemma 3. *Let $f : X \to \mathbb{P}^1_k$ be a nonconstant morphism. Let t be a coordinate on $\mathbb{P}^1_k$. Choose $g \in K$ and put $h = \mathrm{Trace}_{K/k(t)}\, g$. Then for each closed point $P \in \mathbb{P}^1_k$,*

$$\sum_{Q \in f^{-1}(P)} \mathrm{Res}_Q(g\, dt) = \mathrm{Res}_P(h\, dt).$$

Proof. Explicit computation. For a more elegant derivation using a more conceptual definition of the residue, see Tate's paper "Residues of differentials on curves". □

Lemma 4 (Residue theorem)**.** *For s a meromorphic differential on X, the sum of* $\mathrm{Res}_P(s)$ *over all closed points P on X equals* 0.

Proof. For $X = \mathbb{P}^1_k$, this was proved on a homework. The general case reduces to this case via Lemma 3. □

We now use residues to take a closer look at $H^1(X, \mathcal{O}_X(D))$ using Weil's method of *repartitions*. Let η be the generic point of X; we can identify η with $\mathrm{Spec}(K)$ and then view the inclusion $i_\eta : \eta \to X$ as a morphism of schemes. The sheaf $i_{\eta*}\mathcal{O}_\eta$ on X is quasicoherent and assigns each nonempty open set to K, so we have a short exact sequence

$$0 \to \mathcal{O}_X \to i_{\eta*}\mathcal{O}_\eta \to \bigoplus_{P \in X} i_{P*}(K/\mathcal{O}_{X,P}) \to 0$$

where P runs over closed points and $i_P : \mathrm{Spec}(k) \to X$ is the map with image P. Tensor with $\mathcal{O}_X(D)$, which preserves the exact sequence because $\mathcal{O}_X(D)$ is locally free. Then identify $(K/\mathcal{O}_{X,P}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ with $K/\mathcal{O}_X(D)_P$ and take the long exact sequence in cohomology:

$$0 \to k \to K \to \bigoplus_{P \in X} K/\mathcal{O}_X(D)_P \to H^1(X, \mathcal{O}(D)) \to 0.$$

The last zero is $H^1(X, i_{\eta*}\mathcal{O}_\eta \otimes_{\mathcal{O}_X} \mathcal{O}(D))$, which vanishes because the sheaf $i_{\eta*}\mathcal{O}_\eta \otimes_{\mathcal{O}_X} \mathcal{O}(D)$ has surjective restriction maps (i.e., it is *flasque*). We conclude that

$$H^1(X, \mathcal{O}(D)) \cong \mathrm{coker}\left(K \oplus \bigoplus_{P \in X} \mathcal{O}_X(D)_P \to \bigoplus_{P \in X} K\right), \tag{2}$$

where K maps diagonally and the sums over P map term-by-term. Similarly,

$$H^1(X, \omega_X(D)) \cong \mathrm{coker}\left(\omega_{X,\eta} \oplus \bigoplus_{P \in X} \omega_{X,P} \to \bigoplus_{P \in X} \omega_{X,\eta}\right) \tag{3}$$

where $\omega_{X,\eta}$ is the stalk at η, i.e., the space of meromorphic differentials on X. Since Res_P vanishes on $\omega_{X,P}$ and $\mathrm{Res} = \sum_P \mathrm{Res}_P$ vanishes on $\omega_{X,\eta}$ by Lemma 4, we get a well-defined map

$$\mathrm{Res} : H^1(X, \omega_X) \to k.$$

Using Res, we define a bilinear map

$$H^0(X, \omega_X(D)) \times H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \omega_X) \overset{\mathrm{Res}}{\to} k \tag{4}$$

and hence a map

$$H^0(X, \omega_X(D)) \to H^1(X, \mathcal{O}_X(-D))^*, \tag{5}$$

where $*$ denotes the k-linear dual: $M^* = \mathrm{Hom}_k(M, k)$. The key statement will be the following lemma.

Lemma 5. *The bilinear map* (4) *is a perfect pairing; that is, the induced map* (5) *is an isomorphism. In particular, $g = g'$ (by taking $D = 0$).*

It is pretty tricky to prove this directly for a single D; instead, we will prove it for all D simultaneously! Suppose D' is another divisor such that $D \leq D'$. Then on one hand there is an obvious inclusion of $H^0(X, \omega_X(D))$ into $H^0(X, \omega_X(D'))$. On the other hand, from the interpretation of H^1 using (2), there is also an injection $H^1(X, \mathcal{O}_X(-D))^* \to H^1(X, \mathcal{O}_X(-D'))$; or if you prefer, this is the transpose of a map $H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X(-D'))$, which one sees is surjective by taking cohomology on the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X(-D') \to \mathcal{F} \to 0$$

and noticing that $H^1(X, \mathcal{F}) = 0$ because $\mathcal{F}$ is concentrated at finitely many points. In any case, the diagram

$$\begin{tikzcd}
H^0(X, \omega_X(D)) \arrow[r] \arrow[d] & H^1(X, \mathcal{O}_X(-D))^* \arrow[d] \\
H^0(X, \omega_X(D')) \arrow[r] & H^1(X, \mathcal{O}_X(-D'))^*
\end{tikzcd}$$

commute, so the maps (5) combine to give a single map

$$\bigcup_D H^0(X, \omega_X(D)) \to \bigcup_D H^1(X, \mathcal{O}_X(-D))^*. \tag{6}$$

To prove Lemma 5, it is not *a priori* enough to check that (6) is an isomorphism; one must also check the following.

Lemma 6. *For $D \leq D'$, if the image of $s \in H^0(X, \omega_X(D'))$ in $H^1(X, \mathcal{O}_X(-D'))^*$ belongs to $H^1(X, \mathcal{O}_X(-D))^*$, then $s \in H^0(X, \omega_X(D))$.*

Proof. Suppose on the contrary that $s \notin H^0(X, \omega_X(D))$. Then there is a closed point $P \in X$ such that $\mathrm{ord}_P((s) + D) < 0$ but $\mathrm{ord}_P((s) + D') \geq 0$. In particular, $\mathrm{ord}_P(D') > \mathrm{ord}_P(D)$. Choose $f \in K$ so that $\mathrm{ord}_P((s) + (f)) = -1$; then $\mathrm{ord}_P((f) - D) = \mathrm{ord}_P((f) + (s)) - \mathrm{ord}_P((s) + (D)) > -1$, so $f \in \mathcal{O}_X(-D)_P$. Take the class in $H^1(X, \mathcal{O}(-D'))$ defined by the element of $\bigoplus_{P \in X} K$ with f at position P and 0 elsewhere; it then maps to zero in $H^1(X, \mathcal{O}(-D))$. But s maps this class to $\mathrm{Res}_P(fs) \neq 0$, contradiction. □

With this in hand, we are ready to establish duality.

Proof of Lemma 5. We will instead prove that (6) is an isomorphism. This will imply injectivity of (5) directly and surjectivity using Lemma 6.

By writing $K = \bigcup_E H^0(X, \mathcal{O}_X(E))$, we may view both sides of (6) as K-vector spaces and the map as a K-linear transformation. The left side is $\omega_{X,\eta}$, the space of meromorphic differentials, which is of dimension 1 over K. Moreover, the map is injective by Lemma 6: if $s \in H^0(X, \omega_X(D))$ maps to zero in $\bigcup_D H^1(X, \mathcal{O}_X(-D))^*$, then it must belong to $H^0(X, \omega_X(D'))$ for all $D' \leq D$. That is only possible for $s = 0$.

So to get surjectivity of (6), it is enough to check that the right side is of dimension at most 1 over K; that is, any two elements of the right side are linearly dependent. This comes down to an explicit computation using the Riemann inequality, as follows.

Let c_1, c_2 be two elements of the target of (6); we may as well take them to be in $H^1(X, \mathcal{O}_X(-D))^*$ for the same D. Let E be a divisor of some degree n (to be chosen later). If c_1 and c_2 were linearly independent over K, then $(f, g) \mapsto fc_1 + gc_2$ would define an injection of $H^0(X, \mathcal{O}_X(E)) \oplus H^0(X, \mathcal{O}_X(E))$ into $H^1(X, \mathcal{O}_X(-D - E))^*$, so

$$2 \dim_k H^0(X, \mathcal{O}(E)) \leq \dim_k H^1(X, \mathcal{O}_X(-D - E)).$$

If we take n large enough that $n + \deg(D) > 0$, then $H^0(X, \mathcal{O}_X(-D-E))$ is forced to vanish, so by Lemma 2 we have

$$\begin{aligned} \dim_k H^0(X, \mathcal{O}_X(E)) &\geq n + 1 - g' \\ \dim_k H^1(X, \mathcal{O}_X(-D - E)) &= g' - 1 + n + \deg(D). \end{aligned}$$

Thus $g' - 1 + n + \deg(D) \geq 2(n + 1 - g')$, but is a contradiction for n large enough. Hence c_1 and c_2 are linearly dependent over K, completing the proof. □

Lemma 5 plus Lemma 2 together give Riemann-Roch in full.

Theorem 7. *We have $g = g'$ and*

$$\chi_g(D) = \deg(D) + 1 - g'.$$