Math 203b: Algebraic Geometry (UC San Diego, winter 2013)
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Notes on sheaf cohomology and Čech cohomology

In these notes, we explain the relationship between sheaf cohomology and Čech cohomology, with a special emphasis on quasicoherent sheaves on schemes. Throughout these notes, let X be a topological space.

1 Lemmas from homework

Lemma 1. *For any sheaf $\mathcal{F}$ of abelian groups on X, there exists an injective morphism $0 \to \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ an injective sheaf.*

Proof. Homework. □

A sheaf $\mathcal{F}$ of abelian groups on X is *flasque* if for any open sets $U \subseteq V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Lemma 2. *Any injective sheaf is flasque.*

Proof. Homework. □

Lemma 3. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves of abelian groups on X with $\mathcal{F}$ flasque, then*

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is exact for any open subset U of X.

Proof. Homework. □

2 Sheaf cohomology

On an upcoming homework, we will construct the *sheaf cohomology* functors $\mathcal{F} \mapsto H^i(X, \mathcal{F})$ from the category of sheaves of abelian groups to the category of abelian groups. Note that even when the sheaves have more structure, we will always compute sheaf cohomology by ignoring the extra structure. One of the built-in properties of sheaf cohomology (which is basically a corollary of the snake lemma) is the *long exact sequence* property: for any short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

there is a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \stackrel{\delta^0}{\to}$$
$$\to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \stackrel{\delta^1}{\to} \cdots .$$

More precisely, the maps $H^i(X,\mathcal{F}) \to H^i(X,\mathcal{G})$ and $H^i(X,\mathcal{G}) \to H^i(X,\mathcal{H})$ are the ones induced by functoriality, and the *connecting homomorphisms* δ^i are also functorial in the sense that for any commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{H}' \arrow[r] & 0 \end{tikzcd}$$

with exact rows, the diagrams

$$\begin{tikzcd} H^i(X,\mathcal{H}) \arrow[r, "\delta^0"] \arrow[d] & H^{i+1}(X,\mathcal{F}) \arrow[d] \\ H^i(X,\mathcal{H}') \arrow[r, "\delta^0"] & H^{i+1}(X,\mathcal{F}') \end{tikzcd}$$

also commute.

We say $\mathcal{F}$ is *acyclic* if $H^i(X,\mathcal{F}) = 0$ for all $i > 0$. For example, it will follow from the construction that any injective sheaf of abelian groups is acyclic. An even stronger statement is the following.

Lemma 4. *Any flasque sheaf is acyclic.*

Proof. Homework. $\square$

Warning: if one replaces sheaf cohomology with some other left exact functor, then injective sheaves are still acyclic, but flasque sheaves need not be.

3 Čech cohomology

Let us now recall how Čech cohomology is computed. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. The associated *Čech complex*

$$0 \to \check{C}^0(X,\mathcal{F},\mathcal{U}) \to \check{C}^1(X,\mathcal{F},\mathcal{U}) \to \cdots$$

has $\check{C}^j(X,\mathcal{F},\mathcal{U})$ equal to the direct product of the groups $\mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_j})$ for all $i_0,\dots,i_j \in I$. The map $\check{C}^j(X,\mathcal{F},\mathcal{U}) \to \check{C}^{j+1}(X,\mathcal{F},\mathcal{U})$ acts by

$$(s_{i_0,\dots,i_j})_{i_0,\dots,i_j\in I} \mapsto \left(\sum_{k=0}^{j+1}(-1)^k s_{i_0,\dots,\widehat{i_k},\dots,i_j}\right)_{i_0,\dots,i_{j+1}\in I},$$

where the hat means i_k is omitted. The groups $\check{H}^j(X,\mathcal{F},\mathcal{U})$ for $j \geq 0$ are then defined as

$$\check{H}^j(X,\mathcal{F},\mathcal{U}) = \frac{\ker(\check{C}^j(X,\mathcal{F},\mathcal{U}) \to \check{C}^{j+1}(X,\mathcal{F},\mathcal{U}))}{\mathrm{im}(\check{C}^{j-1}(X,\mathcal{F},\mathcal{U}) \to \check{C}^j(X,\mathcal{F},\mathcal{U}))}$$

with the convention that $\check{C}^{-1} = 0$. Note that $\check{H}^0(X, \mathcal{F}, \mathcal{U}) = H^0(X, \mathcal{F})$, the group of global sections, no matter what $\mathcal{U}$ is; but in general, the groups $\check{H}^j(X, \mathcal{F}, \mathcal{U})$ will depend on j.

Warning: it is not true in general that the Čech cohomology groups fit into a long exact sequence, even if one takes the direct limit over all coverings!

We'll say that $\mathcal{F}$ is *$\mathcal{U}$-acyclic* if $\check{H}^j(X, \mathcal{F}, \mathcal{U}) = 0$ for all $j > 0$. We'll say that $\mathcal{F}$ is *strongly Čech-acyclic* if there exist a cofinal set of open coverings $\mathcal{U}$ (i.e., every open covering can be refined to one of the given ones) for each of which $\mathcal{F}$ is $\mathcal{U}$-acyclic. This implies that $\mathcal{F}$ is *Čech-acyclic*, meaning that $\check{H}^j(X, \mathcal{F})$ vanishes for all $j > 0$.

Lemma 5. *If $\mathcal{F}$ is injective, then $\mathcal{F}$ is $\mathcal{U}$-acyclic for every open cover $\mathcal{U}$, and hence strongly Čech-acyclic.*

Proof. See Stacks Project, Lemma 18.11.1, tag 01EP. More on the argument a bit later. □

Lemma 6. *If $\mathcal{F}$ is flasque, then $\mathcal{F}$ is $\mathcal{U}$-acyclic for every open cover $\mathcal{U}$, and hence strongly Čech-acyclic.*

Proof. We will prove that $\check{H}^i(X, \mathcal{F}, \mathcal{U}) = 0$ for $i = 1, \dots, m$ by induction on m, with empty base case $m = 0$. Suppose the claim is given for some m. Given $\mathcal{F}$, use Lemma 1 to form an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

where $\mathcal{G}$ is injective and hence flasque by Lemma 2. Since $\mathcal{F}$ is flasque, by Lemma 3

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is exact for any open subset U of X. This implies on one hand, $\mathcal{H}$ is flasque by the snake lemma, and the other hand, in this case we do get a long exact sequence on Čech cohomology groups!

To prove the claim for $m = 1$, note that surjectivity of $H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$ means that

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to \check{H}^1(X, \mathcal{F}, \mathcal{U}) \to \check{H}^1(X, \mathcal{G}, \mathcal{U}) = 0,$$

the connecting homomorphism $H^0(X, \mathcal{H}) \to \check{H}^1(X, \mathcal{F}, \mathcal{U})$ is zero. Therefore, $\check{H}^1(X, \mathcal{F}, \mathcal{U})$ is trapped between a zero map and a zero object, so it must be zero.

To prove the claim for $m + 1$ given the claim for m, note that $\mathcal{G}$ is already $\mathcal{U}$-acyclic by Lemma 5. Hence

$$0 = \check{H}^m(X, \mathcal{G}, \mathcal{U}) \to \check{H}^m(X, \mathcal{H}, \mathcal{U}) \to \check{H}^{m+1}(X, \mathcal{F}, \mathcal{U}) \to \check{H}^{m+1}(X, \mathcal{G}, \mathcal{U}) = 0$$

is exact and $\check{H}^m(X, \mathcal{H}, \mathcal{U}) = 0$ by the induction hypothesis, so $\check{H}^{m+1}(X, \mathcal{F}, \mathcal{U}) = 0$ also. □

Lemma 7. *If $\check{H}^1(X, \mathcal{F}) = 0$, then for any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ the sequence*

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

is exact.

Proof. As usual, it suffices to check that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective. Given $s \in \mathcal{H}(X)$, there exists an open covering $\mathcal{U}$ of X such that for each $i \in I$, s lifts to a section $t_i \in \mathcal{G}(U_i)$. The differences $t_i - t_j$ then define an element of $\check{H}^1(X, \mathcal{F}, \mathcal{U})$; by hypothesis, we can replace $\mathcal{U}$ by a finer cover so that the element of $\check{H}^1(X, \mathcal{F}, \mathcal{U})$ vanishes. That is, there exist elements $u_i \in \mathcal{F}(U_i)$ with $u_i - u_j = t_i - t_j$; the elements $t_i - u_i$ then glue to a section in $\mathcal{G}(X)$ lifting s. $\square$

Lemma 8. *If $\mathcal{F}|_U$ is Čech-acyclic for each open set U in some basis of the topology of X, then $\mathcal{F}$ is acyclic.*

The proof is quite similar to that of Lemma 6.

Proof. We will prove that $H^i(X, \mathcal{F}) = 0$ for $i = 1, \dots, m$ by induction on m, with empty base case $m = 0$. Suppose the claim is given for some m. Given $\mathcal{F}$, use Lemma 1 to form an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

where $\mathcal{G}$ is injective and hence flasque by Lemma 2. By Lemma 7, taking sections over any open subset in a basis of X gives an exact sequence. By the snake lemma, it follows that $\mathcal{H}$ is flasque and hence Čech-acyclic by Lemma 6.

To prove the claim for $m = 1$, note that by Lemma 3, $H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$ is surjective. Therefore, in the exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) = 0,$$

the connecting homomorphism $H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$ is zero. Therefore $H^1(X, \mathcal{F})$ is trapped between a zero map and a zero object, so it must be zero.

To prove the claim for $m + 1$ given the claim for m, note that

$$0 = H^m(X, \mathcal{G}) \to H^m(X, \mathcal{H}) \to H^{m+1}(X, \mathcal{F}) \to H^{m+1}(X, \mathcal{G}) = 0$$

is exact and $H^m(X, \mathcal{H}) = 0$ by the induction hypothesis, so $H^{m+1}(X, \mathcal{F}) = 0$ also. $\square$

This specializes to the following result (which is not so convenient to prove directly).

Theorem 9. *If X is an affine scheme and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\mathcal{F}$ is acyclic.*

Proof. We proved in class that $\mathcal{F}$ is $\mathcal{U}$-acyclic whenever $\mathcal{U}$ is a finite cover by distinguished open affines. But since X is quasicompact, every covering can be refined by such a covering, so $\mathcal{F}$ is strongly Čech acyclic. By Lemma 8, $\mathcal{F}$ is also acyclic. $\square$

Lemma 10. *Let*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$$

be an exact sequence of sheaves of abelian groups on X such that $\mathcal{F}_i$ is acyclic for all $i \geq 0$ (also called an acyclic resolution *of $\mathcal{F}$). Then for all $i \geq 0$,*

$$H^i(X, \mathcal{F}) \cong \frac{\ker(\mathcal{F}_i(X) \to \mathcal{F}_{i+1}(X))}{\operatorname{im}(\mathcal{F}_{i-1}(X) \to \mathcal{F}_i(X))}$$

with the convention that $\mathcal{F}_{-1} = 0$.

Proof. Put $\mathcal{G}_0 = \mathcal{F}$, then split the exact sequence as

$$\begin{gathered} 0 \to \mathcal{G}_0 \to \mathcal{F}_0 \to \mathcal{G}_1 \to 0 \\ 0 \to \mathcal{G}_1 \to \mathcal{F}_1 \to \mathcal{G}_2 \to 0 \\ 0 \to \mathcal{G}_2 \to \mathcal{F}_2 \to \mathcal{G}_3 \to 0 \\ \vdots \end{gathered}$$

From long exact sequences, we have

$$H^i(X, \mathcal{F}) = H^i(X, \mathcal{G}_0) \cong H^{i-1}(X, \mathcal{G}_1) \cong \cdots \cong H^1(X, \mathcal{G}_{i-1})$$

and exact sequences

$$\begin{gathered} H^0(X, \mathcal{F}_{i-1}) \to H^0(X, \mathcal{G}_i) \to H^1(X, \mathcal{G}_{i-1}) \to 0 \\ 0 \to H^0(X, \mathcal{G}_i) \to H^0(X, \mathcal{F}_i) \to H^0(X, \mathcal{G}_{i+1}) \\ 0 \to H^0(X, \mathcal{G}_{i+1}) \to H^0(X, \mathcal{F}_{i+1}). \end{gathered}$$

This yields the desired result. (One can automate this sort of argument using the theory of *spectral sequences*, but I won't talk more about those here.) □

This statement also has a converse: if $\mathcal{C}$ is a class of sheaves such that the cohomology of every sheaf can be computed using a resolution with terms in $\mathcal{C}$, then every sheaf in $\mathcal{C}$ is acyclic. The proof is simple: for each $\mathcal{F} \in \mathcal{C}$, the sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \cdots$$

with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i = 0$ for $i > 0$ defines a resolution of $\mathcal{F}$, so we must have $H^i(X, \mathcal{F}) = 0$ for $i > 0$!

Lemma 11. *Let $f : X \to Y$ be a continuous map. If $\mathcal{F}$ is acyclic, then so is $f_*\mathcal{F}$.*

Proof. We will prove that $H^i(X, \mathcal{F}) = 0$ for $i = 1, \ldots, m$ by induction on m, with empty base case $m = 0$. Given $\mathcal{F}$, use Lemma 1 to form an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

where $\mathcal{G}$ is injective and hence flasque by Lemma 2. Then

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to 0$$

is again exact. Since $f_*\mathcal{G}$ is also flasque, it is acyclic.

To prove the claim for $m = 1$, note that $H^0(Y, f_*\mathcal{G}) \to H^0(Y, f_*\mathcal{H})$ is surjective, so in the exact sequence

$$0 \to H^0(Y, f_*\mathcal{F}) \to H^0(Y, f_*\mathcal{G}) \to H^0(Y, f_*\mathcal{H}) \to H^1(Y, f_*\mathcal{F}) \to H^1(Y, f_*\mathcal{G}) = 0,$$

the connecting homomorphism $H^0(Y, f_*\mathcal{H}) \to H^1(Y, f_*\mathcal{F})$ is zero. Therefore $H^1(Y, f_*\mathcal{F})$ is trapped between a zero map and a zero object, so it must be zero.

To prove the claim for $m+1$ given the claim for m, note that

$$0 = H^m(Y, f_*\mathcal{G}) \to H^m(Y, f_*\mathcal{H}) \to H^{m+1}(Y, f_*\mathcal{F}) \to H^{m+1}(Y, f_*\mathcal{G}) = 0$$

is exact and $H^m(Y, f_*\mathcal{H}) = 0$ by the induction hypothesis, so $H^{m+1}(Y, f_*\mathcal{F}) = 0$ also. □

Theorem 12. *Let X be a scheme and let $\mathcal{F}$ be a quasicoherent sheaf on X. Let $\mathcal{U}$ be a finite open covering such that each nonempty intersection among the elements of $\mathcal{U}$ is affine. (For instance, if X is separated, this happens if each of the individual opens in $\mathcal{U}$ is affine.) Then*

$$H^i(X, \mathcal{F}) \cong \check{H}^i(X, \mathcal{F}, \mathcal{U}).$$

In particular, $\check{H}^i(X, \mathcal{F}, \mathcal{U})$ does not depend on $\mathcal{U}$ as long as $\mathcal{U}$ satisfies these conditions. A direct proof of this last statement in the case where X is separated and quasicompact appears in Gathmann, Lemma 8.5.2.

Proof. We make a sheafified version of the Čech complex. For $j \geq 0$, let V_j be the disjoint union of the schemes $U_{i_0} \cap \cdots \cap U_{i_j}$ for $i_0, \dots, i_j \in I$, and let $i_j : V_j \to X$ be the map obtained by combining the obvious open immersions. Then put $\check{\mathcal{C}}^j(X, \mathcal{F}, \mathcal{U}) = i_{j*} i_j^* \mathcal{F}$, so that $H^0(X, \check{\mathcal{C}}^j(X, \mathcal{F}, \mathcal{U})) = \check{C}^j(X, \mathcal{F}, \mathcal{U})$. Since $i_j^* \mathcal{F}$ is quasicoherent, it is acyclic by Theorem 9, as then is its pushforward by Lemma 11 (whether or not the pushforward is quasicoherent, which is only guaranteed if $\mathcal{U}$ is a finite cover). So the $\check{\mathcal{C}}^j(X, \mathcal{F}, \mathcal{U})$ form an acyclic resolution, so Lemma 10 tells us what we want. □

As a corollary, we get the following.

Theorem 13. *Let X be a separated scheme and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then*

$$H^i(X, \mathcal{F}) \cong \check{H}^i(X, \mathcal{F}).$$

Proof. Since X is separated, any cover $\mathcal{U}$ by open affines satisfies the hypothesis of Theorem 12, so

$$H^i(X, \mathcal{F}) \cong \check{H}^i(X, \mathcal{F}, \mathcal{U}).$$

Since covers by open affines are cofinal, this proves the claim. □

One can also prove this without assuming X is separated, but it's a bit more complicated. In fact, the equality

$$H^i(X, \mathcal{F}) \cong \check{H}^i(X, \mathcal{F})$$

is true in some generality for topological spaces, but sadly not in complete generality (otherwise things would be much easier here!).

4 δ-functors

A *(covariant) δ-functor* between abelian categories is a sequence of functors T^i plus morphisms $\delta^i : T^i(C) \to T^{i+1}(A)$ for each short exact sequence $0 \to A \to B \to C \to 0$, such that every short exact sequence $0 \to A \to B \to C \to 0$ gives rise to a long exact sequence

$$
\begin{aligned}
0 \to T^0(A) \to T^0(B) \to T^0(C) &\xrightarrow{\delta^0} \\
\to T^1(A) \to T^1(B) \to T^1(C) &\xrightarrow{\delta^1} \cdots .
\end{aligned}
$$

and for any commuting diagram

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

with exact rows, the diagrams

$$
\begin{tikzcd}
T^i(C) \arrow[r, "\delta^0"] \arrow[d] & T^{i+1}(A) \arrow[d] \\
T^i(C') \arrow[r, "\delta^0"] & T^{i+1}(A')
\end{tikzcd}
$$

also commute. In particular, T^0 must be left exact. For example, the sheaf cohomology groups $T^i(\mathcal{F}) = H^i(X, \mathcal{F})$ form a δ-functor.

A *universal δ-functor* is one for which given another δ-functor $(T')^i$ between the same two categories, any natural transformation $f^0 : T^0 \to (T')^0$ of functors extends uniquely to a collection of natural transformations $f^i : T^i \to (T')^i$ which commute with the δ maps. It can be shown directly that if the source category has enough injectives, then the derived functors of a left exact functor have this property, but this also follows from a more general result which can be applied even without having enough injectives.

A functor F is *coeffaceable* if for each object A in its source category, there is a monomorphism $A \to B$ whose image under F is the zero morphism. For example, for $i > 0$, the functor $H^i(X, \cdot)$ is coeffaceable because injective sheaves are acyclic and the category of sheaves of abelian groups has enough injectives.

Theorem 14. *Let T^i be a covariant δ-functor such that T^i is coeffaceable for all $i > 0$. Then T^i is a universal δ-functor.*

In particular, sheaf cohomology forms a universal δ-functor.

Proof. (Taken from Stacks Project, Lemma 10.9.4, tag 010T). Let $(T')^i$ be another δ-functor and let $f^0 : T^0 \to (T')^0$ be a natural transformation of functors. Given $f^0, \ldots, f^n$, we construct f^{n+1} as follows. Given A, make an injection $A \to B$ with $T^{n+1}(A) \to T^{n+1}(B)$ the zero morphism, and complete to a short exact sequence $0 \to A \to B \to C \to 0$. By the long exact sequence, $T^{n+1}(A) \cong \operatorname{coker}(T^n(B) \to T^n(C))$. We then define $t_A^{n+1} : T^{n+1}(A) \to (T')^{n+1}(A)$ as the unique map making the diagram

$$\begin{tikzcd}
\operatorname{coker}(T^n(B) \to T^n(C)) \arrow[r, "t^n"] \arrow[d] & \operatorname{coker}((T')^n(B) \to (T')^n(C)) \arrow[d] \\
T^{n+1}(A) \arrow[r, "t_A^{n+1}"] & (T')^{n+1}(A)
\end{tikzcd}$$

commute. Some annoying details (like functoriality) omitted. $\square$

One may be tempted to use Theorem 14 to compare sheaf cohomology and Čech cohomology, but the latter does not in general define a δ-functor. However, if we view it as a functor on *presheaves* then it does define a δ-functor. This, plus the fact that an injective sheaf of groups is also injective in the category of presheaves, yields Lemma 5.