We have already defined an \emph{open immersion} to be a morphism \( f : Y \to X \) which induces an isomorphism of \( Y \) with an open subset of \( X \). This was easy because an open subset of \( X \) inherits a scheme structure directly from \( X \).

But from the context of varieties, we know we would also like to define \emph{closed subschemes} of a given scheme \( X \). This is harder because it is not obvious how to put a scheme structure on a closed set; just taking \( f^{-1} O_Y \) doesn’t work because we don’t get a locally ringed space. Also, it doesn’t match what we want for varieties: we would like for instance to start with the affine plane \( \text{Spec} K[x, y] \), take the locus where \( x = 0 \), and get the affine line \( \text{Spec} K[y] \).

It turns out there is a good reason why this is subtle: in the category of schemes, there are usually many different “closed subspaces” with the same underlying set! For instance, in the example of the affine plane, we can also form \( \text{Spec} K[x, y] / (x^n) \) for any positive integer \( n \), and this has the same underlying set as \( \text{Spec} K[y] \) but is not isomorphic as a scheme.

In fact, we would like to say that a morphism of affine schemes \( \text{Spec} B \to \text{Spec} A \) corresponds to a closed subspace whenever \( A \to B \) is a surjective morphism of rings.

\textbf{Lemma 1.} Let \( f : Y \to X \) be a morphism of schemes. Then the property “\( Y \times_X \text{Spec} A = \text{Spec} B \) for some \( B \) such that \( A \to B \) is surjective” is a local property of open affine subschemes \( \text{Spec} A \) of \( X \).

\textit{Proof.} It is obvious that this property passes from \( \text{Spec} A \) to \( \text{Spec} A_f \). Thus we need only check that if \( X = \text{Spec} A, f_1, \ldots, f_n \in A \) generate the unit ideal, and \( Y \times_X \text{Spec} A_{f_i} = \text{Spec} B_i \) for some ring \( B_i \) such that \( A_{f_i} \to B_i \) is surjective, then \( Y = \text{Spec} B \) for some ring \( B \) such that \( A \to B \) is surjective.

There are various ways to see this, but one elegant way uses what we know about quasicoherent sheaves. Note that the kernel of a map \( \mathcal{F} \to \mathcal{G} \) of quasicoherent sheaves is again quasicoherent: it locally corresponds to the kernel at the level of modules. (Warning: this is again true for cokernels, but it is not obvious because taking quotients of sheaves involves a sheafification step. We’ll discuss this again shortly.)

Let \( I \) be the sheaf \( \ker(\mathcal{O}_X \to f_* \mathcal{O}_Y) \); by the previous discussion, it is quasicoherent, and hence corresponds to an \( A \)-module \( I \) via the third fundamental theorem of schemes. Again, since kernels between modules and quasicoherent sheaves correspond, the map \( I \to A \) is an inclusion, so \( I \) may be viewed as an ideal of \( A \). Put \( B = A/I \); from the isomorphisms \( Y \times_X \text{Spec} A_{f_i} = \text{Spec} B_i \cong \text{Spec} B_{f_i} \), we may assemble an isomorphism \( Y \cong \text{Spec} B \). \( \square \)

We therefore define a \emph{closed immersion} to be any morphism \( f : Y \to X \) of schemes such that for some (hence any) open covering of \( X \) by affine schemes \( \text{Spec} A \), for each \( A \) we have \( Y \times_X \text{Spec} A = \text{Spec} B \) for some ring \( B \) for which \( A \) \( \to B \) is surjective. (The definition in Hartshorne is slightly different and ultimately equivalent; we will reconcile them a bit later.)
Let us again emphasize the fact that while the image of a closed immersion is indeed a closed subset of $X$, it is not determined by that image. For example, consider the diagram

$$
\text{Spec } K[x, y]/(x) \longrightarrow \text{Spec } K[x, y]/(x^2) \longrightarrow \text{Spec } K[x, y]/(x^3) \longrightarrow \cdots
$$

in which all of the arrows are closed immersions. The first object in the top row corresponds to the “reduced” $y$-axis, whereas the later objects correspond to various “infinitesimally thicker” copies of the $y$-axis.