**Math 203B: Algebraic Geometry**
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**Derived functors**

In preparation for sheaf cohomology, we give a brief overview of the formalism of derived functors. (Note: to avoid left-right confusion, all functors will be covariant; if you want to deal with contravariant functors, flip the source or the target appropriately.)

# 1 Cohomological functors

Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact covariant functor between two abelian categories (say modules over certain rings; remember that abelian groups are themselves modules over the ring $\mathbb{Z}$). A *cohomological functor* (or $\delta$-functor) associated to $F$ is a collection of functors $F^0, F^1, \ldots : \mathcal{C}_1 \to \mathcal{C}_2$ with $F^0 = F$ together with, for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

a family of *connecting homomorphisms*

$$\delta^i : F^i(M_3) \to F^{i+1}(M_1)$$

such that the half-infinite sequence

$$0 \to F^0(M_1) \to F^0(M_2) \to F^0(M_3) \xrightarrow{\delta^0} F^1(M_1) \to F^1(M_2) \to F^1(M_3) \xrightarrow{\delta^1} \cdots$$

is exact. We also want the $\delta^i$ to be functorial for short exact sequences; that is, given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1' & \longrightarrow & M_2' & \longrightarrow & M_3' & \longrightarrow & 0 \end{array}$$

with exact rows, the diagrams

$$\begin{array}{ccc} F^i(M_3) & \xrightarrow{\delta^i} & F^{i+1}(M_1) \\ \downarrow & & \downarrow \\ F^i(M_3') & \xrightarrow{\delta^i} & F^{i+1}(M_1') \end{array}$$

should commute.

## 2 Example: the snake lemma

Here is the simplest example of this phenomenon. Let $R$ be any ring, and let $F : \mathbf{Mod}_{R[T]} \to \mathbf{Mod}_R$ be the functor taking $M$ to $\ker(\times T : M \to M)$. Then we obtain a cohomological functor by taking $F^0 = F$,

$$F^1(M) = \operatorname{coker}(\times t : M \to M),$$

and $F^i(M) = 0$ for $i \geq 2$. We then need to define, for each short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

a single connecting homomorphism $\delta : \ker(M_3) \to \operatorname{coker}(M_1)$ making the sequence

$$0 \to \ker(M_1) \to \ker(M_2) \to \ker(M_3) \stackrel{\delta}{\to} \operatorname{coker}(M_1) \to \operatorname{coker}(M_2) \to \operatorname{coker}(M_3) \to 0$$

exact. The existence of this map is a consequence of the well-known *snake lemma*, which is found in many references. To summarize, the map $\delta$ is constructed as follows: given $x \in \ker(M_3)$, it lifts to some $y \in M_2$ which is not necessarily in the kernel. However, $Ty \in M_2$ maps to zero in $M_3$ and hence lifts to some $z \in M_1$, and we want $\delta(x)$ to equal the image of $z$ in $\operatorname{coker}(M_1)$. This leaves a number of loose ends:

- Why is $\delta$ well-defined?
- Why is it a morphism of $R$-modules?
- Why is the composition $\ker(M_2) \to \ker(M_3) \to \operatorname{coker}(M_1)$ zero?
- Why is the composition $\ker(M_3) \to \operatorname{coker}(M_1) \to \operatorname{coker}(M_2)$ zero?
- Why is the sequence exact at $\ker(M_3)$?
- Why is the sequence exact at $\operatorname{coker}(M_1)$?

None of these is individually hard to check, but the totality of them is a bit more than I want to explain here. (Optional exercise: find Dick Gross at UCSD and ask him to tell you a fun story about the snake lemma!)

## 3 Universal cohomological functors

Reminder: a morphism $F \to F'$ of functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ consists of, for each object $X$ of $\mathcal{C}_1$, a morphism $F(X) \to F'(X)$ in $\mathcal{C}_2$, chosen in such a way that whenever $X \to Y$ is a morphism in $\mathcal{C}_1$, the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F'(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F'(Y) \end{array}$$

commutes.

A cohomological functor $(F^i, \delta^i)$ associated to $F$ is *universal* if given any morphism $F \to F'$ of functors and any cohomological functor $(F'^i, \delta'^i)$ associated to $F'$, there exist a unique family of morphisms $F^i \to F'^i$ such that for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

the diagram

$$\begin{array}{ccc} F^i(M_3) & \xrightarrow{\delta^i} & F^{i+1}(M_1) \\ \downarrow & & \downarrow \\ F'^i(M_3) & \xrightarrow{\delta'^i} & F^{i+1}(M_1) \end{array}$$

commutes. This being a universal property, when such a thing exists, it is unique up to unique isomorphism.

# 4 Example: the snake lemma revisited

Define $F : \mathbf{Mod}_{R[T]} \to \mathbf{Mod}_R$ and the associated cohomological functor as before. I claim it is universal: given another cohomological functor $(F'^i, \delta'^i)$ and a morphism $F \to F'$, we get a unique morphism $F^1 \to F'^1$ commuting with $\delta^0$.

The way to see this is to observe that given any $M \in \mathbf{Mod}_{R[T]}$, I can find an injective morphism $M \to N$ in $\mathbf{Mod}_{R[T]}$ such that $\operatorname{coker}(\times T : N \to N) = 0$. For example, take $N$ to be an infinite product $M \times M \times \cdots$ with multiplication by $T$ taking $(m_0, m_1, \dots)$ to $(Tm_0 + m_1, Tm_1 + m_2, \dots)$. Then the map $M \to N$ taking $m$ to $(m, 0, 0, \dots)$ is $T$-equivariant, hence a morphism in $\mathbf{Mod}_{R[T]}$, and any $(m_0, m_1, \dots)$ is the image of $(0, m_0, m_1 - Tm_0, m_2 - Tm_1 + T^2 m_0, \dots)$.

Now start with the short exact sequence

$$0 \to M \to N \to N/M \to 0$$

and form the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F(M) & \longrightarrow & F(N) & \longrightarrow & F(N/M) & \xrightarrow{\delta^0} & F^1(M) & \longrightarrow & F^1(N) = 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & F'(M) & \longrightarrow & F'(N) & \longrightarrow & F'(N/M) & \xrightarrow{\delta'^0} & F'^1(M) & & \end{array}$$

and note that there is a unique way to fill in the dashed arrow. (If you just checked this by hand, you have engaged in the art of *diagram chasing*.)

So now we at least know that the maps $F^1(M) \to F'^1(M)$ are unique; it is also clear that this defines a morphism of functors (since the construction of $N$ is itself functorial in

$M$). What remains is to check that this definition gives the desired commutativity

$$\begin{array}{ccc} F(M_3) & \xrightarrow{\delta} & F^i(M_1) \\ \downarrow & & \downarrow \\ F'(M_3) & \xrightarrow{\delta'} & F^i(M_1) \end{array}$$

for *any* short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

and not just the special ones I used in the definition. You can do this by diagram chasing, but maybe easier is to stare at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \oplus N_1 & \longrightarrow & M_3' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & N_1/M_1 & \longrightarrow & 0 \end{array}$$

with exact rows, where $N_1$ is to $M_1$ as $N$ is to $M$ above. From this diagram, we see that the diagrams

$$\begin{array}{ccccc} F(M_2) & \longrightarrow & F(M_3) & \xrightarrow{\delta} & F^1(M_1) \\ \downarrow & & \downarrow & & \| \\ F(M_2 \oplus N_1) & \longrightarrow & F(M_3') & \xrightarrow{\delta} & F^1(M_1) \\ \uparrow & & \uparrow & & \| \\ F(N_1) & \longrightarrow & F(N_1/M_1) & \xrightarrow{\delta} & F^1(M_1) \end{array} \qquad \begin{array}{ccccc} F'(M_2) & \longrightarrow & F'(M_3) & \xrightarrow{\delta'} & F'^1(M_1) \\ \downarrow & & \downarrow & & \| \\ F'(M_2 \oplus N_1) & \longrightarrow & F'(M_3') & \xrightarrow{\delta'} & F'^1(M_1) \\ \uparrow & & \uparrow & & \| \\ F'(N_1) & \longrightarrow & F'(N_1/M_1) & \xrightarrow{\delta'} & F'^1(M_1) \end{array}$$

commute. This implies that the commutativity statement about the $\delta$ in the first row follows from the corresponding statement in the second row, which in turns follows from the corresponding statement in the third row. (This last step requires a bit of thought: given $x \in F(M_3')$, we would be happy if $x$ itself lifted to $F(N_1/M_1)$, but it may not. However, by looping around the square, I can find $y \in F(M_3')$ which does lift to $F(N_1/M_1)$ for which $\delta(x-y) = 0$. This $x - y$ lifts to $z \in F(M_2 \oplus N_1)$. Now on one hand, mapping $x - y$ along $F(M_3') \to F'(M_1) \to F'^1(M_1)$ must give zero because we get zero at the first step; on the other hand, mapping $x - y$ along $F(M_3') \to F'(M_3) \to F'(M_1)$ is the same as mapping $z$ along $F(M_2 \oplus N_1) \to F'(M_2 \oplus N_1) \to F'(M_3') \to F'^1(M_1)$ because $F \to F'$ is a morphism of functors, and $F'(M_2 \oplus N_1) \to F'(M_3') \to F'^1(M_1)$ is the zero map.)

## 5 Some general results

**Theorem 1.** *If $\mathcal{C}_1 = \mathbf{Mod}_R$ for some ring $R$, then every left exact functor admits a universal cohomological functor associated to it.*

**Theorem 2.** *If $\mathcal{C}_1 = \mathbf{Mod}_R^{\mathrm{opp}}$ for some ring $R$, then every left exact functor admits a universal cohomological functor associated to it.*

In order to imitate the previous strategy, one should identify, for each $M$, an exact sequence

$$0 \to M \to N \to N/M \to 0$$

for which $F^i(N) = 0$ for all $i > 0$. (In fact, we could get by knowing only that $F^i(M) \to F^i(N)$ is the zero map for all $i > 0$, but I digress.) If we could do this, then we would know that $F^1(M) = \operatorname{coker}(F(N) \to F(N/M))$; we would also know that $F^i(N/M) \to F^{i+1}(M)$ is an isomorphism for $i > 0$. So if we had another sequence

$$0 \to N/M \to N' \to N'/(N/M) \to 0$$

for which $F^i(N') = 0$ for all $i > 0$, then we would get $F^2(M) \cong F^1(N/M) \cong \operatorname{coker}(F(N') \to F(N'/(N/M))$. To encapsulate, if $F$ is a cohomological functor and

$$0 \to M \to N^0 \to N^1 \to \cdots$$

is an exact sequence for which $F^i(N^j) = 0$ for all $i > 0$, then

$$\begin{aligned} F^0(M) &\cong \ker(F(N^0) \to F(N^1)) \\ F^1(M) &\cong \ker(F(N^1) \to F(N^2))/\operatorname{image}(F(N^0) \to F(N^1)) \\ F^2(M) &\cong \ker(F(N^2) \to F(N^3))/\operatorname{image}(F(N^1) \to F(N^2)) \end{aligned}$$

and so on. That is, the groups $F^i(M)$ are the *cohomology groups* of the complex

$$0 \to F(N^0) \to F(N^1) \to \cdots ;$$

in particular, all of these but the 0-th one vanish if $F$ is exact.

Of course, all of this is a bit circular; we can't use this as the definition without knowing what the $F^i$ are, in order to check that $F^i(N^j) = 0$ for all $i > 0$. But can we come up with an *a priori* set of candidates for the $N_j$?

In fact we can, because of the following observation: even if $F$ is only left exact, it is forced to preserve exactness of certain sequences. Namely, suppose

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence which is *split*, that is, there exists a morphism $M_3 \to M_2$ such that $M_3 \to M_2 \to M_3$ is the identity, or equivalently there exists a morphism $M_2 \to M_1$ such that

$M_1 \to M_2 \to M_1$ is the identity. Then this additional arrow-theoretic condition is preserved by $F$, which forces

$$0 \to F(M_1) \to F(M_2) \to F(M_3) \to 0$$

to remain exact (namely, if $F(M_3) \to F(M_2) \to F(M_3)$ is the identity, then $F(M_2) \to F(M_3)$ must be surjective).

What this suggests is that if $N$ is an object for which *every* short exact sequence

$$0 \to N \to * \to * \to 0$$

splits, perhaps we should expect that $F^i(N) = 0$ for all $i > 0$. (This is a much more ambitious guess than what we actually know, which is only that the connecting homomorphisms into $F^1(N)$ all vanish, but bear with me.) This suggests in turn that we try defining the $F^i$ to be the cohomology groups of the complex

$$0 \to F(N^0) \to F(N^1) \to \cdots$$

for some exact sequence

$$0 \to M \to N^0 \to N^1 \to \cdots$$

where each $N^i$ has the splitting property.

In the case of $\mathbf{Mod}_R^{\mathrm{opp}}$, every free module has the splitting property. In fact, we can say more: if $N$ is free, then for any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ & & \downarrow & \swarrow & \\ & & N & & \end{array}$$

in $\mathbf{Mod}_R^{\mathrm{opp}}$ with the first row exact, there exists (but not uniquely) a way to fill in the dashed arrow. That is, $N$ is an *injective object* in $\mathbf{Mod}_R^{\mathrm{opp}}$, a/k/a a *projective object* in $\mathbf{Mod}_R$. Since every module is a quotient of a free module, the category $\mathbf{Mod}_R^{\mathrm{opp}}$ *has enough injectives*: every object admits a monomorphism into an injective object.

Meanwhile, $\mathbf{Mod}_R$ also has enough injectives, but this is harder to see because it is harder to get a hold of explicit injective objects. The case $R = \mathbb{Z}$ is a bit easier: an abelian group is injective if and only if it is divisible.

(Aside: once an abelian category has enough injectives, the injective property for an object becomes equivalent to the splitting condition.)

To summarize, if the origin category has enough injectives, we define $F^i$ to be the cohomology groups of the complex

$$0 \to F(N^0) \to F(N^1) \to \cdots$$

for some exact sequence

$$0 \to M \to N^0 \to N^1 \to \cdots$$

where each $N^i$ is injective (the existence of said sequence, called an *injective resolution*, being guaranteed by the enough injectives condition). This leaves a lot of unanswered questions:

- Why is this well-defined, i.e., independent of the choice of the resolution?
- Why is this functorial? (It is arguably easier to do this before well-definedness. More precisely, one checks that given a morphism $M \to M'$ and fixed resolutions of $M$ and $M'$, one gets functoriality maps that don't depend on any choices; then well-definedness comes from taking $M = M'$.)
- Where do the connecting homomorphisms come from, and why are they functorial? (Ultimately, this reduces to the snake lemma.)
- Why is this universal? (Given everything else, this is similar to the example.)

I don't want to say more than this, because we have reached the point where it is more profitable for you to read and think about these topics on your own. See Wikipedia for basics, then follow the links (also try the Stacks Project).