Math 203B: Algebraic Geometry
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Hilbert polynomials

1 Euler characteristics and Hilbert polynomials

Let K be a field (not necessarily algebraically closed). Let $j : X \to \mathbb{P}^d_K$ be a closed immersion. Let $\mathcal{F}$ be a coherent (quasicoherent locally finitely generated) sheaf on X. By the previous results, it makes sense to compute the number

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_K H^i(X, \mathcal{F});$$

this is called the *Euler characteristic* of $\mathcal{F}$ on X. Note that this is additive in short exact sequences: if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence of coherent sheaves, then

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}).$$

Theorem 1. *There is a polynomial $P(T) \in \mathbb{Q}[T]$ depending on X and $\mathcal{F}$ such that $\chi(X, \mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$.*

This polynomial is called the *Hilbert polynomial* of $\mathcal{F}$. In case $\mathcal{F} = \mathcal{O}_X$, it is also called the *Hilbert polynomial* of X itself.

Note that for n sufficiently large, we have $\chi(X, \mathcal{F}(n)) = \dim_K H^0(X, \mathcal{F}(n))$ because all of the other terms vanish. That is, the *Hilbert function*

$$n \mapsto \dim_K H^0(X, \mathcal{F}(n))$$

of $\mathcal{F}$ agrees with the Hilbert polynomial for n large; the higher cohomology groups in a sense explain the discrepancy for n small.

(Fun aside: there is an analogous situation in combinatorics involving counting lattice points in dilates of a polytope, which can be explained by algebraic geometry to the extent that a certain combinatorial duality property is a consequence of the Serre duality theorem we will state later. Look up the terms *Ehrhart polynomial* and then *toric varieties* on Wikipedia to get started.)

2 Proof of Theorem 1

There are a variety of ways to prove Theorem 1. In all of them, one makes the usual reduction to the case $X = \mathbb{P}^d_K$.

A quick way is to use a famous result of commutative algebra called the *Hilbert syzygy theorem*, which states that for any finitely generated module M over a polynomial ring $K[x_0, \dots, x_d]$ over a field, there exists a resolution

$$0 \to F_d \to \cdots \to F_0 \to M \to 0$$

in which $F_0, \dots, F_d$ are finite free modules. The point is that one gets to stop after F_d; for a more general noetherian ring, you can use exclusively finite free modules but you will typically have to go on forever. Typical example: for $R = K[x]/(x^2)$,

$$\stackrel{\times x}{\to} R \stackrel{\times x}{\to} R \to K \to 0.$$

How is this relevant here? We know that we can form an exact sequence

$$\cdots \to \mathcal{F}_d \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0$$

of sheaves in which each $\mathcal{F}_i$ has the form $\mathcal{O}(n_i)^{\oplus m_i}$ for some integers m_i, n_i. For each of those, we know that the theorem holds because

$$\chi(\mathbb{P}^d_K, \mathcal{O}(n)) = \binom{n+d}{d} = \frac{(n+1)\cdots(n+d)}{d!}.$$

If we had only finitely many terms, we would then have

$$\chi(\mathbb{P}^d_K, \mathcal{F}) = \chi(\mathbb{P}^d_K, \mathcal{F}_0) - \chi(\mathbb{P}^d_K, \mathcal{F}_1) + \cdots$$

and be done.

The trick is to notice that thanks to the syzygy theorem, the sheaf image$(\mathcal{F}_d \to \mathcal{F}_{d-1})$ is already finite locally free! This comes down to the algebraic statement: if you have a module M over a ring admitting a finite free resolution

$$0 \to F_d \to \cdots \to F_0 \to M \to 0$$

then for any other resolution

$$\cdots \to F'_1 \to F'_0 \to M \to 0$$

the module $F''_d = \text{image}(F'_d \to F'_{d-1})$ is itself projective. (Argue by induction on d.) In other words, if M admits one finite free resolution (or equivalently, one finite *projective* resolution), then any other projective resolution can be truncated to the same length.

3 Another proof of Theorem 1

That proof is elegant, but (besides requiring proof of Hilbert's syzygy theorem) doesn't give a lot of insight into how the polynomial $P(T)$ relates to the geometry of X and $\mathcal{F}$. A more insightful argument can be obtained by the following inductive process.

Again, assume $X = \mathbb{P}^d_K$. Define the *support* of $\mathcal{F}$ to be the set of $x \in \mathbb{P}^d_K$ for which $\mathcal{F}_x \neq 0$. For $\mathcal{F}$ a coherent sheaf, this is always a closed subset of $\mathbb{P}^d_K$ (the equality $\mathcal{F}_x = 0$ depends on the vanishing of finitely many local generators, which then immediately propagate to a neighborhood). For example, if we had started with $\mathcal{F} = \mathcal{O}_X$ for some other X, then $j_*\mathcal{O}_X$ has support equal to the image of the closed immersion $j : X \to \mathbb{P}^d_K$.

This time, we will argue by induction on $\dim X$. If you prefer, you may as well assume K is algebraically closed, since base extension on the underlying field won't change any dimensions. (Rest to be added later.)

Viewing $\operatorname{Supp} \mathcal{F}$ as a closed subvariety of X, we may find a hyperplane H which does not contain any irreducible component of $\operatorname{Supp} \mathcal{F}$. (Explicitly, think about the *dual projective space* whose K-rational points correspond to these hyperplanes; for each component, the hyperplanes not containing that component form a nonempty Zariski open subspace. So the intersection of these is again nonempty.) Form an exact sequence

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{H} \to 0$$

where the middle map is multiplication by a defining equation of H. (Note that $\mathcal{G} \neq 0$ because we don't know that $\mathcal{F}$ is flat as a module over $\mathcal{O}$.) At points not in H, the map $\mathcal{F}(-1) \to \mathcal{F}$ defines an isomorphism of stalks; consequently, we have

$$\operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{H} \subseteq H \cap \operatorname{Supp} \mathcal{F}.$$

By the induction hypothesis, we see that

$$\chi(X, \mathcal{F}(n)) - \chi(X, \mathcal{F}(n-1))$$

is a polynomial in n, as then is $\chi(X, \mathcal{F})$ by elementary algebra.

A corollary of this argument is that $\deg P = \dim \operatorname{Supp} \mathcal{F}$. If we call this number m, then $m!$ times the leading coefficient of P is a positive integer, called the *degree* of $\mathcal{F}$.