**Math 203B: Algebraic Geometry**
**UCSD, winter 2016, Kiran S. Kedlaya**
**Line bundles on curves and the Riemann-Roch theorem**

Throughout this lecture, let $K$ be an algebraically closed field and let $C$ be a *curve* over $K$, by which I will mean a smooth irreducible projective variety of dimension 1 over $K$ (or rather, the associated scheme).

# 1 A note on local rings

First off, $C$ has a generic point $\eta$, and the local ring $\mathcal{O}_{C,\eta}$ equals the function field $K(C)$. (For instance, if $C = \mathbb{P}^1_K$ with coordinate $x$, then $K(C) = K(x)$.)

For each closed point $P \in C$, the local ring $\mathcal{O}_{C,P}$ is a one-dimensional noetherian local ring. Note that the map

$$\mathfrak{m}_{C,P}/\mathfrak{m}_{C,P}^2 \cong \Omega_{\mathcal{O}_{C,P}/K}/\mathfrak{m}_{C,P}\Omega_{\mathcal{O}_{C,P}/K}, \qquad t \mapsto dt$$

is an isomorphism of $\kappa(P)$-vector spaces; since by smoothness $\Omega_{\mathcal{O}_{C,P}/K}$ is free of rank one, it follows that $\mathfrak{m}_{C,P}/\mathfrak{m}_{C,P}^2$ must be a one-dimensional vector space over $\kappa(P)$. Consequently, if we choose an element $t_p \in \mathfrak{m}_{C,P} - \mathfrak{m}_{C,P}^2$ (i.e., a *uniformizer* of $\mathcal{O}_{C,P}$), then $dt_p$ is a free generator of $\Omega_{\mathcal{O}_{C,P}/K}$. In fact, $\mathcal{O}_{C,P}$ must be a discrete valuation ring.

# 2 Divisors and degrees

A *divisor* on $C$ is a formal $\mathbb{Z}$-linear combination of closed points. For example, for any nonzero rational function $f \in K(C)$ (where $K(C)$ is the function field of $C$, a/k/a the local ring $\mathcal{O}_{C,\eta}$ where $\eta$ is the generic point of $C$), we define a divisor $(f) = \sum_P \operatorname{ord}_P(f) \cdot (P)$ where $\operatorname{ord}_P f$ is the order of vanishing of $f$ at $P$. More precisely, if $t_P$ is a uniformizer of $\mathcal{O}_{C,P}$, then $\operatorname{ord}_P(f)$ is the integer $m$ such that $ft_P^{-m}$ is a unit in $\mathcal{O}_{C,P}$.

This concept extends to line bundles. If $\mathcal{L}$ is a line bundle on $C$, a *rational section* of $\mathcal{L}$ is an element $s \in \Gamma(U, \mathcal{L})$ for some nonempty open subset $U$ of $C$. For $s$ a nonzero rational section, we define a divisor $(s) = \sum_P \operatorname{ord}_P(s) \cdot (P)$ where $\operatorname{ord}_P(s)$ is the unique integer $m$ for which $st_P^{-m}$ is a generator of $\mathcal{L}_P$.

The *degree* of a divisor is the sum of its coefficients. A divisor occurring as $(f)$ for some $f \in K(C)$ is called a *principal divisor*.

**Theorem 1.** *Every principal divisor has degree* 0.

*Proof.* For $C = \mathbb{P}^1_K$ this is clear because we can factor any nonzero $f \in K(C)$ as a product of powers of linear polynomials, and the polynomial $x - \alpha$ has divisor $(\alpha) - (\infty)$. The general case reduces to this using the existence of a finite surjective morphism $C \to \mathbb{P}^1_K$; see homework. □

As a corollary, we see that for $s$ a nonzero rational section of $\mathcal{L}$, the quantity $\deg(s)$ depends only on $\mathcal{L}$, so we write it as $\deg(\mathcal{L})$ and call it the *degree* of $\mathcal{L}$.

# 3 Aside: line bundles from divisors

We just used line bundles to make divisors, but one can also go the other way. Given a divisor $D = \sum_P D_P \cdot (P)$ on $C$, we can form a line bundle $\mathcal{O}(D)$ whose sections on a nonempty open subset $U$ of $C$ are the rational functions $f \in K(C)$ such that either $f = 0$ or $\mathrm{ord}_P(f) + D_P \geq 0$ for all $P \in U$. Note that the rational section corresponding to $1 \in K(C)$ then has divisor precisely $D$.

# 4 Statement of Riemann-Roch

**Theorem 2** (Riemann-Roch)**.** *For every line bundle $\mathcal{L}$ on $C$, there is a canonical perfect pairing*

$$H^0(C, \mathcal{L}) \times H^1(C, \Omega \otimes \mathcal{L}^{-1}) \to K.$$

*In particular, the two vector spaces have the same dimension.*

Note that the case $\mathcal{L} = \mathcal{O}_C$ of this statement is already interesting: it says that there is a canonical isomorphism

$$H^1(C, \Omega) \cong K.$$

When $K = \mathbb{C}$, there is a way to prove this using complex analysis: a meromorphic differential on a Riemann surface has a well-defined *residue* at each point, and the sum of these over all points equals 0 (by the Cauchy integral formula). The map $H^1(C, \Omega) \to K$ is then defined as follows: if we cover $C$ with two open subsets $U_1, U_2$ and then specify an element of $H^1(C, \Omega)$ with a form $\omega \in \Gamma(U_1 \cap U_2, \Omega)$, we then map it to the sum of its residues.

To extend this proof to general $K$, note that (by a previous exercise) one can formally define the *residue* of a meromorphic differential form $\omega$ at a point $P \in C$ by choosing a uniformizer $t_P$, writing the completion of the local ring $\mathcal{O}_{C,P}$ as a power series ring $K[\![t_P]\!]$, then expanding $\omega$ as a formal series

$$\omega = \sum_{n=-N}^{\infty} a_n t_P^n dt_P$$

and taking the residue to be $a_{-1}$. The key point is then to show that the sum of residues of a meromorphic differential always equals 0. This again can be reduced to the case $C = \mathbb{P}^1_K$ using a finite morphism. In that case, one can write the differential as $(f(x)/g(x))\,dx$ with $f$ and $g$ polynomials; for any fixed degrees of $f$ and $g$, the vanishing of the sum of differentials is some formal polynomial identity over $\mathbb{Z}$ in the coefficients of $f$ and $g$. But this identity must hold over $\mathbb{C}$ by the analytic argument from above, so it must in fact be true identically.