With the Riemann-Roch theorem in hand, we study some constructions leading to curves of particular genera. Again, let $k$ be an algebraically closed field.

## 1 Hyperelliptic curves

A **hyperelliptic curve** is a curve $C$ admitting a finite morphism $f : C \to \mathbb{P}^1_k$ of degree 2. For example, every affine curve of the form $y^2 = P(x)$ in $\mathbb{A}^2_k$ extends to a hyperelliptic curve, with $x$ defining the map to $\mathbb{P}^1_k$. It will follow from the Riemann-Hurwitz formula (see below) that if $\text{deg} P = d$, then the genus of $C$ equals $\lceil \frac{d^2}{2} - 1 \rceil$; in particular, this proves that the genus of a curve can take any nonnegative integer value.

For example, if $\text{deg} P = 1$ one obviously gets $\mathbb{P}^1_k$ by eliminating $x$; if $\text{deg} P = 2$ one has a conic section; if $\text{deg} P = 3$ one gets a smooth cubic curve in $\mathbb{P}^2_k$. For $\text{deg} P > 3$, this affine curve does not extend smoothly in $\mathbb{P}^2_k$, so the genus formula for smooth plane curves does not apply!

## 2 Riemann-Hurwitz formula

**Theorem 1.** Let $f : C_1 \to C_2$ be a finite morphism of degree $n$. (In positive characteristic, we have to assume that $f$ is separable, i.e., that $k(C_1)/k(C_2)$ is not only finite but also separable as a field extension.) Then

$$2g(C_1) - 2 = n(2g(C_2) - 2) + \deg R$$

where $R$ is a divisor associated to $(f^*\Omega_{C_2/k})^\vee \otimes \Omega_{C_1/k}$.

More precisely, we have an exact sequence

$$0 \to f^*\Omega_{C_2/k} \to \Omega_{C_1/k} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a sheaf supported at finitely many points; we may canonically (i.e., not just up to equivalence) take $R$ to be the *ramification divisor*, i.e., the divisor consisting of the points of the support of $\mathcal{F}$, each point $P$ occurring with multiplicity equal to the length of $\mathcal{F}_P$ as a module over $\mathcal{O}_{C,P}$. (Note: the formula now proves itself!)

For example, if $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is the map $x \mapsto z = x^2$, $P \in C_1$ is the point $x = 0$, and $Q \in C_1$ is the point $z = 0$, then $\Omega_{C_1/k,Q}$ is generated by $dz$, which pulls back to $d(x^2) = 2xdx$. If $k$ is not of characteristic 2, then this means that $R$ contains $P$ with multiplicity 1; similarly, the point $P' \in C_1$ where $x = \infty$ is also contained in $R$ with multiplicity 1. With this, the arithmetic works out:

$$2g(C_1) - 2 = -2 = 2(-2) + 2 = n(2g(C_2) - 2) + \deg R.$$
3 Characteristic zero versus characteristic \( p \)

In characteristic zero, it is very easy to compute the divisor \( R \). Namely, if \( P \in C_1 \) mapsto \( Q \in C_2 \), a uniformizer \( t_Q \in \mathcal{O}_{C_2,Q} \) pulls back to an element of the form \( t_P^m u \) for \( t_P \in \mathcal{O}_{C_1,P} \) a uniformizer, \( m \) a positive integer, and \( u \in \mathcal{O}_{C_1,P} \) a unit. We then have

\[
f^*(dt_Q) = \left( mt_P^{m-1}udt_P + t_P^m \frac{du}{dt_P} \right) dt_P.
\]

Since \( m \neq 0 \) in \( k \), \( R \) has multiplicity \( m - 1 \) at \( P \).

Another way to interpret this is that \( R \) consists of the “missing preimages”: most points of \( C_2 \) have exactly \( n \) distinct preimages in \( C_1 \), but a few fall short, and

\[
\deg(R) = \sum_{Q \in C_2} (n - \#f^{-1}(Q)).
\]

This can also be used to give a topological proof of Riemann-Hurwitz over \( \mathbb{C} \): If \( U \) is the complement in \( C_2 \) of the image of the support of \( R \), then \( f^{-1}(U) \to U \) is everywhere \( n \)-to-1, so we have an equality of topological Euler characteristics:

\[
\chi(f^{-1}(U)) = n\chi(U).
\]

Since Euler characteristics are additive over writing a topological space as a union of an open subspace and its complement, and a point has Euler characteristic 1, this yields the proof. (Another way to interpret this is as a proof that the genus in Riemann-Roch coincides with the topological genus: we know this for \( \mathbb{P}^1 \mathbb{C} \), and this derivation implies that both genera transform the same way under finite morphisms.)

This still works in characteristic \( p \) if none of the integers \( m \) is divisible by \( p \); in this case we say \( f \) is tamely ramified (e.g., the squaring map example when \( p \neq 2 \)). If this fails (and \( f \) is separable), we say \( f \) is wildly ramified; these often arise from Artin-Schreier field extensions (see homework).

4 Linear systems

If \( \mathcal{L} \) is a line bundle on \( C \) and \( V \) is a subspace of \( H^0(C, \mathcal{L}) \) of dimension \( n \), we’ve seen in a previous homework that we can attempt to define a map \( C \to \mathbb{P}^{n-1}_k \) using the sections of \( V \); this works provided that the divisors of the nonzero elements of \( V \) have no common point. (Classical terminology: the projectivization of \( V \), or the corresponding collections of divisors, is called a linear system on \( C \). A common point in the divisors is called a base point or basepoint. If there are no base points, we say \( V \) is basepoint-free.)

So let’s try this using the canonical sheaf \( \Omega \), taking \( V \) to be the whole space of sections (which has dimension \( g \)). If \( g = 0 \), then \( V = 0 \) and this completely fails. If \( g = 1 \), then \( V \) is a one-dimensional space; it is basepoint-free since any section has degree \( 2g - 2 = 0 \), but we just get a map to a point.
This gets more interesting once \( g \) gets up to 2. In this case, \( V \) is a two-dimensional space, so we potentially are getting a map \( C \to \mathbb{P}^1_k \), at least provided that there is no basepoint. (In fact, the canonical linear system is always basepoint-free for \( g \geq 2 \); see homework.) The degree of this map can be interpreted as the degree of any nonzero divisor in the linear system, which in this case is \( 2g - 2 = 2 \). So in fact, the one construction we know of curves of genus 2, namely as hyperelliptic curves, is in fact the only way that they can occur!