Math 203B: Algebraic Geometry
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Sections of quasicoherent sheaves

We continue with the discussion of quasicoherent sheaves, and in particular to what extent they are generated by global sections.

1 Surjections of quasicoherent sheaves

First, a quick definition from homological algebra. A sequence of maps

$$A \to B \to C$$

between abelian groups is *exact* if the following conditions hold.

- The composition $A \to B \to C$ equals zero. (This means that the sequence is a *complex*.)
- The induced inclusion
$$\operatorname{image}(A \to B) \to \ker(B \to C)$$
is in fact an isomorphism.

For sequences longer than three terms, we impose the same condition on every three consecutive terms. For instance, a *short exact sequence*

$$0 \to A \to B \to C \to 0$$

means that A is isomorphic to a subgroup of B, the quotient by which is isomorphic to C. For any category admitting a faithful forgetful functor to abelian groups (e.g., $\mathbf{Mod}_R$ for some ring R), we make these definitions on the level of underlying abelian groups.

Note that the first condition is preserved by applying any functor, but the second condition is not. A functor that preserves the second condition is said to be *exact*; it is easy shown that this happens if and only if the functor preserves short exact sequences. For example, if $R \to S$ is a *flat* ring homomorphism (e.g., a localization), then the functor $\bullet \otimes_R S : \mathbf{Mod}_R \to \mathbf{Mod}_S$ is exact.

It is more common to have only a one-sided version of the exact condition. For example, a functor which preserves exactness of sequences of the form

$$0 \to A \to B \to C$$

is said to be *left exact* (and similarly for *right exact*).

We say that a sequence of maps of sheaves is *exact* if for each point, the associated sequence of stalks is exact. It is not hard to see that the global sections functor is left exact, using the fact that $\mathcal{F}(X) \to \sqcup_{x \in X} \mathcal{F}_x$ is injective. However, it is not in general right exact; in particular, surjectivity of a morphism of sheaves is *not* in general characterized by surjectivity of maps of global sections.

So in general, if

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is an exact sequence of sheaves, we only get an exact sequence

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \to ???$$

and we can ask whether there is a way to "fill in" the gap on the right. We will use sheaf cohomology to answer this question later; for the moment, let me prove a result that rules out this pathology in particular case of present interest, and also gives the flavor of what we will want to do more generally.

Theorem 1. *Let $X = \operatorname{Spec} R$ be an affine scheme. Then the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ from quasicoherent sheaves on X to $\mathbf{Mod}_R$ is exact.*

Proof. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence. Taking global sections yields a sequence

$$0 \to M \to N \to P$$

of R-modules, and we wish to check that $N \to P$ is surjective. Since localization morphisms of rings are flat, if we write $P' = \operatorname{coker}(M \to N)$ and let $\mathcal{G}'$ be the associated sheaf, then

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G}' \to 0$$

is exact. But then $\mathcal{G} \cong \mathcal{G}'$, so $P = \mathcal{G}(X) = \mathcal{G}'(X) = P'$ and we win. $\square$

In the previous proof, I didn't really need to know that $\mathcal{G}$ is quasicoherent, but I did need this about $\mathcal{E}$ and $\mathcal{F}$. However, one can do a bit better.

Theorem 2. *Let $X = \operatorname{Spec} R$ be an affine scheme. Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of sheaves in which $\mathcal{E}$ is quasicoherent. Then

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \to 0$$

is exact.

Proof. Again, let

$$0 \to M \to N \to P$$

be the sequence obtained by taking global sections. What we know is that given an element $p \in P$, there exist elements $f_1, \dots, f_k \in R$ generating the unit ideal and lifts $n_i \in N_{f_i}$ lifting

p; however, the elements n_i, n_j need not coincide in $N_{f_i f_j}$. On the other hand, they do both map to p in $P_{f_i f_j}$, so the difference $n_i - n_j$ lifts uniquely from $N_{f_i f_j}$ to $M_{f_i f_j}$. (Note that we are using freely the fact that localization is an exact functor.)

At this point, it would be enough to produce elements $m_i \in M_{f_i}$ such that $m_i - m_j = n_i - n_j$: namely, we could then replace the lifts n_i with $n_i - m_i$ to get new lifts of p (they are still lifts because $M \to N \to P$ is exact) which do in fact glue on overlaps.

On HW2, we've seen that the sequence

$$\prod_i M_{f_i} \to \prod_{i,j} M_{f_i f_j} \to \prod_{i,j,k} M_{f_i f_j f_k}$$

is exact, where the first map is the usual $(m_i) \mapsto (m_i - m_j)$ and the second map is $(m_{i,j}) \mapsto (m_{j,k} - m_{i,k} + m_{i,j})$. (If you didn't read the solution, localize at an arbitrary prime ideal and sort things out by hand.) The middle term in this sequence contains $(n_i - n_j)_{i,j}$ which maps to zero (because M_* injects into N_*), so it arises from something on the left. But this is exactly what we needed! □

2 Quasicoherent sheaves on projective spaces

Let's now look at the projective space $X = \mathbb{P}^n_R$ for some ring R. We have seen that $\mathcal{O}(k)$ has "many global sections" if $k > 0$, but not if $k \leq 0$. Here is a much stronger form of that statement.

For $\mathcal{F}$ a quasicoherent sheaf on $\mathbb{P}^n_R$, define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$; this operation is called *twisting* by $\mathcal{O}(n)$. It can be undone by twisting by $\mathcal{O}(-n)$.

Theorem 3 (Serre). *Let R be a ring. Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathbb{P}^d_R$ for some $d \geq 0$ which is locally finitely generated (i.e.,* coherent *under the definition of Hartshorne). Then there exists $n_0 \in \mathbb{Z}$ such that for each $n \geq n_0$, $\mathcal{F}(n)$ is generated by finitely many global sections (i.e., there exists a surjective morphism $\mathcal{O}^{\oplus c} \to \mathcal{F}(n)$ of quasicoherent sheaves for some c depending on n).*

Proof. For $i = 0, \dots, d$, $M_i = \mathcal{F}(D_+(x_i))$ is a finitely generated module over $R[x_0/x_i, \dots, x_d/x_i]$. Choose a finite set of module generators of M_i. It would be enough to show that for each generator s in this set, we can find an integer n such that $x_i^n s$, viewed as a section of $\mathcal{F}(n)$ over $D_+(x_i)$, extends to a section of $\mathcal{F}(n)$ over all of $\mathbb{P}^d_R$; namely, the same would be true for any larger n, so for some suitably large n we can represent generators of $\mathcal{F}(n)(D_+(x_i))$ for all i using global sections.

To begin with, for each j, we may restrict s to $\mathcal{F}(D_+(x_i x_j)) = (M_i)_{x_j/x_i} = (M_j)_{x_i/x_j}$. For suitably large n, $x_i^n s$ lifts from $x_i^n (M_j)_{x_i/x_j}$ to some element $s_j \in x_i^n M_j$.

However, these lifts may not agree on overlaps. No problem: $s_j - s_k \in \mathcal{F}(D_+(x_j x_k)$ must be killed by some power of $x_i/x_j = (x_i/x_j)(x_j/x_k)$ (the latter factor being a unit), so by raising n suitably I can force the difference to become zero. □

This result also promotes formally to closed subschemes of projective space. To see why, note that for any closed immersion $j : Y \to X$ of schemes, if $\mathcal{F}$ is a quasicoherent sheaf on Y, then $f_*\mathcal{F}$ is a quasicoherent sheaf on X: locally we have a surjective map $R \to R/I$ of rings and an R/I-module M, and we are simply viewing M as an R-module. Moreover, if $\mathcal{F}$ is locally finitely generated, then so is $j_*\mathcal{F}$.

Consequently, if $j : X \to \mathbb{P}^d_R$ is a closed immersion and $\mathcal{F}$ is a quasicoherent locally finitely generated sheaf on X, if we define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}} j^*\mathcal{O}(n)$, then there exists $n_0 \in \mathbb{Z}$ such that for each $n \geq n_0$, $\mathcal{F}(n)$ is generated by finitely many global sections.

3 Preview

Here is a related result, but we are not ready to finish its proof just yet.

Theorem 4. *Assume that the ring R is noetherian. Let $j : X \to \mathbb{P}^d_R$ be a closed immersion. Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of coherent sheaves on X. Then there exists $n_0 \in \mathbb{Z}$ such that for each $n \geq n_0$, the sequence

$$0 \to \mathcal{E}(n)(X) \to \mathcal{F}(n)(X) \to \mathcal{G}(n)(X) \to 0$$

is again exact.

Partial proof. We may again reduce to the case $X = \mathbb{P}^d_R$; only the surjectivity of $\mathcal{F}(n)(X) \to \mathcal{G}(n)(X)$ is at issue. Again, any global section can be lifted on $D_+(x_i)$ for each i, and the differences lift uniquely from $\mathcal{F}(n)(D_+(x_i x_j))$ to $\mathcal{G}(n)(D_+(x_i x_j))$. So it would be enough to know that

$$\prod_i \mathcal{E}(n)(D_+(x_i)) \to \prod_{i,j} \mathcal{E}(n)(D_+(x_i x_j)) \to \prod_{i,j,k} \mathcal{E}(n)(D_+(x_i x_j x_k))$$

is exact for n large. $\square$

To fill in the rest of this proof, we will need to describe sheaf cohomology. More on that shortly!