Math 203B: Algebraic Geometry
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Projective subschemes

1 Pullback of quasicoherent sheaves

Let $f : Y \to X$ be a morphism of schemes. Recall that we have a direct image functor f_* from arbitrary sheaves (say of abelian groups) on Y to sheaves on X. This takes quasicoherent sheaves to quasicoherent sheaves: for example, if $Y = \operatorname{Spec} S$, $X = \operatorname{Spec} R$, and $\mathcal{F} = \tilde{M}$ for some $M \in \mathbf{Mod}_S$, then $f_*\mathcal{F} = \tilde{N}$ for N the restriction of scalars of M from S to R (i.e., it's a copy of M but now viewed in $\mathbf{Mod}_R$ instead of $\mathbf{Mod}_S$).

By contrast, the adjoint functor f^{-1} does not take quasicoherent sheaves on X to quasicoherent sheaves on Y. One must replace it with the functor

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y,$$

where $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is adjoint to the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ from the definition of a morphism of locally ringed spaces. For example, if $Y = \operatorname{Spec} S$, $X = \operatorname{Spec} R$, and $\mathcal{F} = \tilde{M}$ for some $M \in \mathbf{Mod}_R$, then $f^*\mathcal{F} = \tilde{N}$ for $N = M \otimes_R S \in \mathbf{Mod}_S$.

The functors f_* and f^* on quasicoherent sheaves are often called *pushforward* and *pullback*. They again form an adjoint pair with f^* on the left, f_* on the right.

2 Quasicoherent sheaves and line bundles

Let X be a scheme and let $\mathcal{F}$ be a sheaf of modules on V. Recall that the property "$\mathcal{F}|_{\operatorname{Spec}(R)} \cong \tilde{M}$ for $M = \mathcal{F}(\operatorname{Spec} R)$" is a local property in the sense of the affine communication lemma; this gave us the definition of a quasicoherent sheaf.

Suppose now that $\mathcal{F}$ is indeed quasicoherent. It will be shown in HW4 that the following are also local properties:

- $\mathcal{F}(\operatorname{Spec} R)$ is a finitely generated R-module;
- $\mathcal{F}(\operatorname{Spec} R)$ is a finitely generated locally free R-module of rank n (where n is a fixed positive integer). (This doesn't work if we drop "locally".)

A quasicoherent sheaf for which $\mathcal{F}(\operatorname{Spec} R)$ is always finitely generated and locally free of rank 1 is commonly called a *line bundle* on X. That is because there is an equivalence of categories between such objects and *geometric line bundles*, the latter being pairs $(\pi : Y \to X, e : X \to Y)$ of morphisms of schemes where $\pi \circ e = \mathrm{id}_X$, such that for some open covering of X by open affines $U_i = \operatorname{Spec}(R_i)$, $Y \cong \mathbb{A}^1_{R_i}$ with π being the map $R_i \to R_i[t]$ and e being the map $R_i[t] \to R_i$ taking t to 0. Pictorially, Y is a "family of one-dimensional vector spaces parametrized by X" and e is the "zero section" picking out the origin in each vector space.

3 Line bundles and graded rings

Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a graded ring. A *graded module* over S is an S-module M of the form $\bigoplus_{n\in\mathbb{Z}} M_n$ where $S_{n_1}M_{n_2} \subseteq M_{n_1+n_2}$ for all n_1, n_2. Any graded module M gives rise to a quasicoherent sheaf $\tilde{M}$ on $\operatorname{Proj} S$ where

$$\tilde{M}(D_+(f)) = M_{f,0}.$$

A key example is given by the shifted modules $S(k)$, where

$$S(k)_n = S_{n+k};$$

let $\mathcal{O}(k)$ be the corresponding sheaf on $\operatorname{Proj} S$.

Suppose now that S_1 generates S_+, which implies that the sets $D_+(f)$ for $f \in S_1$ cover $\operatorname{Proj} S$. In this case, $\mathcal{O}(k)(D_+(f))$ is the free module of rank 1 generated by f, so $\mathcal{O}(k)$ is a line bundle on $\operatorname{Proj} S$. In the key example $S = R[x_0, \dots, x_d]$, we have

$$\mathcal{O}(k)(\mathbb{P}^k_R) = S_k \qquad \text{(i.e., homogeneous polynomials of degree } k\text{).}$$

4 Projective schemes

Let $j : X \to \mathbb{P}^d_R$ be a closed immersion. Then the formula

$$S_k = (j^*\mathcal{O}(1))(X)$$

defines a graded ring S and a map $R[x_0, \dots, x_d] \to S$. The ring map is not necessarily surjective in every degree (think about a large disjoint union of points), but it is surjective in all sufficiently large degrees (this will be shown in a subsequent lecture).

Conversely, if X is a scheme over $\operatorname{Spec} R$ and $\mathcal{F}$ is a line bundle on X, one can ask whether $\mathcal{F}$ occurs as $j^*\mathcal{O}(1)$ for some closed immersion $j : X \to \mathbb{P}^d_R$. This is already an important question in the context of varieties, and will motivate our study of sheaf cohomology.