Math 203B (Algebraic Geometry), UCSD, winter 2016
Solutions for problem set 1

1. (a) Yes, the map is injective: every holomorphic function on an open disc is represented by its Taylor series.

   (b) No, the map is not injective: the function e^{-1/x^2} is C^∞ but its Taylor series is identically zero.

2. Suppose that $s_1, s_2 \in \mathcal{F}(U)$ maps to the same element $\prod_{x \in U} \mathcal{F}_x$. For each $x \in X$, we can find an open neighborhood U_x of x in U such that the restrictions of s_1 and s_2 to U_x coincide. Since the sets U_x form a covering of U, we deduce from the sheaf axiom that $s_1 = s_2$.

   For an exmaple of a presheaf where this fails, take for instance $\mathcal{F}$ to be the sheaf on $\mathbb{R}$ such that

   $$\mathcal{F}(U) = \begin{cases} \mathbb{R} & U = \mathbb{R} \\ \{0\} U \neq \mathbb{R}. \end{cases}$$

   Then $\mathcal{F}_x = 0$ for all $x \in \mathbb{R}$, so the map $\mathcal{F}(\mathbb{R}) \to \prod_{x \in U} \mathcal{F}_x$ cannot be injective.

3. A left adjoint to the forgetful functor **Ab** $\to$ **Set** is given by the functor taking a set S to be the free abelian group on generators indexed by the elements of S. Similarly, a left adjoint to the forgetful functor **Ring** $\to$ **Set** is the functor taking the set S to the polynomial ring over $\mathbb{Z}$ with variables indexed by the elements of S.

4. The generic point (0) has a unique preimage (0). The point (p) has one preimage if $p = 2$ or $p \equiv 3 \pmod 4$, and two preimages if $p \equiv 1 \pmod 4$.

5. The closed points correspond to individual real numbers and to conjugate pairs of nonreal complex numbers.

6. We may write U as the union of the distinguished open subsets $D(2)$, $D(x)$, which intersect in $D(2x)$; we have

   $$D(2) = \operatorname{Spec} \mathbb{Z}[x][1/2], \qquad D(x) = \operatorname{Spec} \mathbb{Z}[x][1/x], \qquad D(2x) = \operatorname{Spec} \mathbb{Z}[x][1/2, 1/x].$$

   The sections $\mathcal{O}_X(U)$ comprise the intersection of the first two rings inside the third one, which is exactly $\mathbb{Z}[x]$.

   If U were affine, it would then equal $\operatorname{Spec} \mathbb{Z}[x]$. By a theorem from class, the inclusion $U \to X$ would then arise from a map $\mathcal{O}_X(X) \to \mathcal{O}_U(U)$. But the source and the target of this map are $\mathbb{Z}[x]$; by restricting to $D(2)$, we see that the map $\mathbb{Z}[x] \to \mathbb{Z}[x]$ we get is the identity map. But $U \to X$ is not an isomorphism, contradiction.

7. We claim that the quotient bijects to $\operatorname{Spec}(R)$. Note that if $R \to F_1, R \to F_2$ are equivalent, then $\ker(R \to F_1) = \ker(R \to F_2)$ are the same prime ideal of R; this gives the map to $\operatorname{Spec}(R)$.

To check that the map is surjective, note that every prime ideal $\mathfrak{p}$ arises as the kernel of $R \to \mathrm{Frac}(R/\mathfrak{p})$.

To check that the map is injective, suppose that $\ker(R \to F_1) = \ker(R \to F_2) = \mathfrak{p}$. Put $F_0 = \mathrm{Frac}(R/\mathfrak{p})$; then F_0 maps to F_1 and F_2, necessarily injectively (field homomorphisms are always injective). Take the ring $F_1 \otimes_{F_0} F_2$ and quotient by any maximal ideal; we then get a field F_3 and homomorphisms $F_1 \to F_3, F_2 \to F_3$ such that the compositions $R \to F_1 \to F_3, R \to F_2 \to F_3$ coincide.