**Math 203B (Algebraic Geometry), UCSD, winter 2016**
**Problem Set 1 (due Wednesday, January 13)**

Solve the following problems, and turn in the solutions to *four* of them. Advance warning: some problems in later sets will refer to earlier problems, so if you don't succeed in solving some problems, make sure to read the posted solutions!

General note: you are welcome to look things up as you try to solve the exercises, as long as you cite your sources. Typical sources: Gathmann's notes, Hartshorne, Atiyah-Macdonald, Stacks Project, Wikipedia, PlanetMath, MathOverflow. (Google is typically *not* a source; cite the originating website.) Likewise, you are welcome to collaborate with other students in the course, as long as you cite these collaborations. (When appropriate, break this down by individual problem.)

If you plan to submit typed solutions, I encourage you to submit them through SageMathCloud. To do this, create a free account at `http://cloud.sagemath.com/` and email me with your username; you will then see a course in your account with upload folders for individual assignments.

Throughout this problem set, you may use without comment the fact that for $R$ any ring, the global sections of the structure sheaf on $\mathrm{Spec}(R)$ equal $R$ (whether or not I get to this in class before Wednesday).

1. (a) Take $X = \mathbb{C}$ and let $\mathcal{F}$ be the sheaf of holomorphic functions. Is the Taylor series map from the stalk $\mathcal{F}_0$ to the power series ring $\mathbb{C}[\![z]\!]$ injective? Why or why not?

   (b) Take $X = \mathbb{R}$ and let $\mathcal{F}$ be the sheaf of $C^\infty$ functions. Is the Taylor series map from the stalk $\mathcal{F}_0$ to the power series ring $\mathbb{R}[\![x]\!]$ injective? Why or why not?

2. Let $\mathcal{F}$ be a sheaf of sets on a topological space $X$. Prove that for every open set $U$, the map
$$\mathcal{F}(U) \mapsto \prod_{x \in U} \mathcal{F}_x, \qquad s \mapsto (s_x)_{x \in U}$$
is injective; also give an example of a presheaf for which this fails.

3. Read about *adjoint functors* on Wikipedia, then describe:

   (a) a left adjoint to the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$;

   (b) a left adjoint to the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$.

   These examples illustrate the common role of adjoint functors as "promotion" and "restriction" operations between categories.

4. Compute the cardinality of each fiber of the map $\mathrm{Spec}\,\mathbb{Z}[i] \to \mathrm{Spec}\,\mathbb{Z}$, using (without proof) standard facts from elementary number theory.

5. Describe the closed points of the topological space $\mathrm{Spec}\,\mathbb{R}[x]$.

6. Let $U$ be the subspace of $X = \operatorname{Spec} \mathbb{Z}[x]$ obtained by removing the closed point $(2, x)$. Compute the sections of the structure sheaf of $X$ on $U$, and use the result to see that $U$ is not affine. (Hint: just like for the affine plane minus a point, you can cover $U$ with two open affine subspaces whose intersection is again open affine.)

7. Let $R$ be a ring. Let $S$ be the "collection" of ring homomorphisms $R \to F$ where $F$ is a field. Declare two such homomorphisms $R \to F_1, R \to F_2$ to be equivalent if there exist field homomorphisms $F_1 \to F_3, F_2 \to F_3$ such that the compositions $R \to F_1 \to F_3, R \to F_2 \to F_3$ coincide. Prove that the quotient of $S$ by this equivalence relation bijects to a set, and identify this set explicitly.