Math 203B (Algebraic Geometry), UCSD, winter 2016
Solutions for problem set 2

1. (a) Let $\underline{\mathcal{B}}$ be the subcategory of $\underline{X}$ consisting of the open sets in $\underline{\mathcal{B}}$ (so again morphisms are inclusions). Then the category of sheaves (say of sets) on X is equivalent to the category of contravariant functors $F : \mathcal{B} \to \mathbf{Set}$ such that for any $U \in \mathcal{B}$ and any covering of U by subsets $U_i \in \mathcal{B}$ (indexed by any set I), $F(U)$ maps bijectively to the set of $(s_i) \in \prod_{i \in I} F(U_i)$ such that for all $i, j \in i$, s_i and s_j have the same image in $F(U_i \cap U_j)$.

   (b) For F a sheaf specified on $\mathcal{B}$ and $x \in X$, we define the stalk F_x again as the direct limit of $\mathcal{F}(U)$ as U runs over elements of $\mathcal{B}$ containing x. Let F' be the sheaf such that $F'(U)$ consists of the maps $s : U \to \sqcup_{x \in X} F_x$ such that $s(x) \in F_x$ for all $x \in U$, and for some covering $\{U_i\}$ of U by elements of $\mathcal{B}$, $s|_{U_i}$ is induced by an element of $F(U_i)$. Then the map $F_x \to F'_x$ is an isomorphism for each x, because elements of $\mathcal{B}$ containing x are cofinal among all open neighborhoods, so the direct limit can be computed just using them. It follows that the functor $F \mapsto F'$ provides a quasi-inverse to the restriction functor from sheaves to sheaves specified on $\mathcal{B}$.

2. Cover the Riemann sphere S with the two open subsets $U_1 = S - \{0\}$ and $U_2 = S - \{\infty\}$. Put $V_1 = \operatorname{Spec} \mathbb{C}[z]$ and $V_2 = \operatorname{Spec} \mathbb{C}[z^{-1}]$; using a theorem from lecture, the maps $\mathbb{C}[z] \to \mathcal{O}(U_1)$, $\mathbb{C}[z] \to \mathcal{O}(U_2)$ define morphisms $U_1 \to V_1, U_2 \to V_2$ which overlap in a morphism $U_1 \cap U_2 \to V_1 \cap V_2$. They thus define a morphism $S \to \mathbb{P}^1_{\mathbb{C}}$ of locally ringed spaces which by the Nullstellensatz is a bijection (but not a homeomorphism) of S with the closed points of $\mathbb{P}^1_{\mathbb{C}}$.

3. (a) The map $A_i \to A_i[f_j^{-1}]$ defines a map $\operatorname{Spec}(A_i[f_j^{-1}]) \to \operatorname{Spec}(A_i) = X_i$. The image of this map contains only primes not containing f_j, so lies in $X_i \cap X_j$. The map $\operatorname{Spec}(A_i[f_j^{-1}]) \to X_i \cap X_j$ is a homeomorphism: its inverse is the continuous map $\mathfrak{p} \mapsto \mathfrak{p}[f_j^{-1}]$.

   (b) To simplify notation, we only check that $A_{f_1} \to A_1$ is an isomorphism. As directed, we start with the exact sequence

$$0 \to A \to \prod_{i=1}^n A_i \to \prod_{i,j=1}^n A_{ij}$$

   for $A_{ij} = \mathcal{O}_X(X_i \cap X_j)$. Inverting f_1 preserves exactness, so we have another exact sequence

$$0 \to A_{f_1} \to \prod_{i=1}^n A_{i,f_1} \to \prod_{i,j=1}^n A_{ij,f_1}.$$

   Using (a), we may rewrite this as

$$0 \to A_{f_1} \to \prod_{i=1}^n \mathcal{O}_X(X_1 \cap X_i) \to \prod_{i,j=1}^n \mathcal{O}_X(X_1 \cap X_i \cap X_j)$$

from which it follows that $A_{f_1} \cong \mathcal{O}_X(X_1) = A_1$.

(c) Using (b), we get a ring map $A \to A_{f_i} \cong A_i$ and hence a morphism $X_i \cong \operatorname{Spec}(A_{f_i}) \to \operatorname{Spec}(A)$ of schemes. These maps agree on overlaps, so they define a morphism $X \to \operatorname{Spec}(A)$ of schemes. To see that this is an isomorphism, it suffices to check locally on A. But $f_1, \dots, f_n$ generate the unit ideal in A, so the distinguished opens $D(f_i)$ form a cover, and the restriction to $D(f_i)$ is the isomorphism $X_i \cong \operatorname{Spec}(A_i)$.

4. The last map in the sequence

$$0 \to M \to \prod_{i=1}^{n} M_{f_i} \to \prod_{i,j=1}^{n} M_{f_i f_j}$$

was defined to take $(s_i)_{i=1}^n$ to $(s_i - s_j)_{i,j=1}^n$ (where the restriction maps have been left implicit). The last map in the extended sequence

$$0 \to M \to \prod_{i=1}^{n} M_{f_i} \to \prod_{i,j=1}^{n} M_{f_i f_j} \to \prod_{i,j,k=1}^{n} M_{f_i f_j f_k}$$

can be taken to send $(s_{ij})_{i,j=1}^n$ to $(s_{ij} - s_{ik} + s_{jk})_{i,j,k=1}^n$. Similarly, to extend the sequence, one maps

$$(s_{i_1 \cdots i_k})_{i_1,\dots,i_k=1}^n \text{ to } \left(\sum_{j=0}^{k} (-1)^j s_{i_0 \cdots \widehat{i_j} \cdots i_k} \right)_{i_0,\dots,i_k=1}^n,$$

where the hat means omit that index.

To see that this sequence is indeed exact, as usual we localize at an arbitrary prime ideal, which effectively means we may assume that $f_1 = 1$. Let C_k be the k-th term in the sequence, indexing so that $C_0 = M$. Let $d_k : C_k \to C_{k+1}$ be the map in the sequence, so that $d_{k+1} \circ d_k = 0$. Let $h_k : C_{k+1} \to C_k$ be the map taking

$$(s_{i_0 \cdots i_k})_{i_0,\dots,i_k=1}^n \text{ to } (s_{1 i_1 \cdots i_k})_{i_1,\dots,i_k=1}^n,$$

using the identification $M_{f_1 f_{i_1} \cdots f_{i_k}} \cong M_{f_{i_1} \cdots f_{i_k}}$. Then $h_k \circ d_k + d_{k-1} \circ h_{k-1} = \mathrm{id}_{C_k}$ (this is an example of a *chain homotopy*). Now if $x \in C_k$ satisfies $d_k(x) = 0$, then

$$x = \mathrm{id}_{C_k}(x) = (h_k \circ d_k + d_{k-1} \circ h_{k-1})(x) = d_{k-1}(h_{k-1}(x)),$$

so x is in the image of C_{k-1}; this proves exactness.

This can be done purely in the language of commutative algebra, but we indicate the following proof in order to illustrate ideas from the lectures so far. Suppose first that $X = \operatorname{Spec}(R)$ can be written as the disjoint union of two nonempty open subsets U_1, U_2. Then there is a section $e_1 \in \mathcal{O}_X(X)$ which restricts to 1 on U_1 and 0 on U_2, and a section $e_2 \in \mathcal{O}_X(X)$ which restricts to 1 on U_2 and 0 on U_1. We proved in class that the natural map $R \to \mathcal{O}_X(X)$ is an isomorphism. To check that e_1, e_2 satisfy $e_1 + e_2 = 1$, $e_1^2 = e_1$, $e_2^2 = e_2$, it is enough to check this at the level of sections, which we may do on U_1 and U_2 separately.

Suppose next that e_1, e_2 are nonzero idempotents which add up to 1. Then $V(e_1), V(e_2)$ are closed subsets of X; they are disjoint because e_1 and e_2 generate the unit ideal, and they cover X because $e_1 e_2 = e_1(1 - e_1) = e_1 - e_1^2 = 0$. Finally, $V(e_1)$ is nonempty: otherwise, e_1 would have to generate the unit ideal, so we could find $f \in R$ with $e_1 f = 1$; but then $e_1 = e_1^2 f = e_1 f = 1$ and so $e_2 = 1 - e_1 = 0$, a contradiction. Similarly, $V(e_2)$ is nonempty, so they form a partition of X into two nonempty closed sets.

5. (a) Let R be a discrete valuation ring with fraction field K. Then $\operatorname{Spec}(R) = \{0, \mathfrak{p}\}$ for $\mathfrak{p}$ the maximal ideal of R; the unique closed point is $\mathfrak{p}$, so $\{0\}$ is an open subset containing no closed points.

   (b) Let R be a finitely generated algebra over a field K. To prove that the closed points of $\operatorname{Spec}(R)$ are dense, it suffices to check that for every $f \in R$, if the distinguished open subset $D(f)$ is nonempty, then $D(f)$ contains a maximal ideal. Suppose to the contrary that every maximal ideal of R contains f. Let I be the nilradical of R. By the Nullstellensatz as in Stacks Project, Tag 00FV, every radical ideal of R is the intersection of the maximal ideals containing it. Applying this to I, we see that $f \in I$, so f is nilpotent; consequently, $D(f)$ is empty.

   (a) Cover $\mathbb{P}^1_{\mathbb{Z}}$ by the open sets $U_1 = \operatorname{Spec} \mathbb{Z}[t]$, $U_2 = \operatorname{Spec} \mathbb{Z}[t^{-1}]$. Then for any morphism $f : \operatorname{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$, we have $f^{-1}(U_1) = V_1, f^{-1}(U_2) = V_2$ for some covering of $\operatorname{Spec}(\mathbb{Z})$ by two open subsets V_1, V_2. Recall that each open subset of $\operatorname{Spec}(\mathbb{Z})$ is either empty or the complement of finitely many maximal ideals; in particular, it is necessarily affine. We thus have $V_1 = \operatorname{Spec} \mathbb{Z}[1/N_1], V_2 = \operatorname{Spec} \mathbb{Z}[1/N_2]$ for some nonnegative integers N_1, N_2 which are coprime.

   Suppose for the moment that $N_1, N_2 > 0$. The maps $V_1 \to U_1, V_2 \to U_2$ correspond to ring maps $\mathbb{Z}[t] \to \mathbb{Z}[1/N_1]$, $\mathbb{Z}[t^{-1}] \to \mathbb{Z}[1/N_2]$ which induce the same map $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[1/(N_1 N_2)]$. That is, t and t^{-1} map to elements of $\mathbb{Q}$ which are reciprocals of each other with respective denominators dividing N_1, N_2. Since N_1, N_2 could in principle be arbitrary, we can in fact realize any nonzero rational number as the image of t.

   The excluded case $N_1 = 0$ corresponds to the map $\mathbb{Z}[t^{-1}] \to \mathbb{Z}$ taking t^{-1} to 0, and vice versa. We conclude that the set of maps is in fact $\mathbb{Q} \cup \{\infty\}$; in other words, it is the same as the set of maps $\operatorname{Spec}(\mathbb{Q}) \to \mathbb{P}^1_{\mathbb{Q}}$!