1. (a) Give a precise mathematical formulation of the following informal statement: for 
    \( X \) a topological space and \( \mathcal{B} \) a basis of open subsets of \( X \), the category of sheaves
    on \( X \) is equivalent to the category of “sheaves specified on \( \mathcal{B} \)”.

    (b) Prove the statement you made in (a).

2. Exhibit a morphism in the category of locally ringed spaces from the Riemann sphere,
    equipped with the sheaf of holomorphic functions, to the scheme \( \mathbb{P}^1_{\mathbb{C}} \)
    which is surjective on closed points. (Optional: show also that this morphism factors through
    the variety-theoretic \( \mathbb{P}^1_{\mathbb{C}} \).

3. Let \( X \) be a scheme and put \( A = \mathcal{O}_X(X) \). Let \( f_1, \ldots, f_n \in A \) be elements which
    generate the unit ideal. For \( i = 1, \ldots, n \), let \( X_i \) be the open subscheme of \( X \) consisting of
    those points \( x \) for which \( f_i \) does not belong to the maximal ideal of the local ring \( \mathcal{O}_{X,x} \).
    Suppose that \( X_i \) is affine for \( i = 1, \ldots, n \), and put \( A_i = \mathcal{O}_X(X_i) \).

    (a) For \( i, j = 1, \ldots, n \), prove that the open subscheme \( X_i \cap X_j \) of \( X \) is isomorphic to
        \( \text{Spec}(A_i[f_j^{-1}]) \).

    (b) Prove that the natural map \( A_{f_i} \to A_i \) is an isomorphism. Hint: start with an
        exact sequence
        \[
        0 \to A \to \bigoplus_{i=1}^n A_i \to \bigoplus_{i,j=1}^n A_{ij}
        \]
        for \( A_{ij} = \mathcal{O}_X(X_i \cap X_j) \), then invert \( f_i \).

    (c) Prove that \( X \) is isomorphic to \( \text{Spec}(A) \).

4. Here’s a fact we’ll use soon in the construction of sheaf cohomology. Let \( M \) be a
    module over a ring \( R \). Let \( f_1, \ldots, f_n \) be elements of \( R \) which generate the unit ideal.
    We showed in lecture for \( M = R \) (but the general case is similar) that there is an exact sequence
    \[
    0 \to M \to \bigoplus_{i=1}^n M_{f_i} \to \bigoplus_{i,j=1}^n M_{f_if_j}.
    \]
    Show that this extends to an exact sequence
    \[
    0 \to M \to \bigoplus_{i=1}^n M_{f_i} \to \bigoplus_{i,j=1}^n M_{f_if_j} \to \bigoplus_{i,j,k=1}^n M_{f_if_jf_k} \to \cdots,
    \]

1
where the definition of the additional terms and maps is left for you to figure out. (It might help to try the case \( n = 3 \) first, then look for the general pattern.)

5. Let \( R \) be a nonzero ring. Prove that the following conditions are equivalent.

   (a) The space \( \text{Spec}(R) \) is disconnected: that is, it is the disjoint union of two open-closed proper subsets.

   (b) There exist nonzero elements \( e_1, e_2 \) of \( R \) with \( e_1 + e_2 = 1 \) which are idempotent, i.e., \( e_1^2 = e_1, e_2^2 = e_2 \).

   Hint: use the fact that \( R \) is isomorphic to the ring of global sections of the structure sheaf.

6. (a) Give an example of a scheme whose closed points are not dense. The Internet can help!

   (b) Show that if \( X = \text{Spec}(R) \) where \( R \) is a finitely generated algebra over a field, then the closed points of \( X \) are dense for the Zariski topology.

7. Let \( S \) be the set of morphisms \( \text{Spec}(\mathbb{Z}) \to \mathbb{P}^1_\mathbb{Z} \). Describe \( S \), and show that in particular \( S \neq \mathbb{Z} \cup \{\infty\} \).