Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 3 (due Wednesday, January 27)

Solve the following problems, and turn in the solutions to *four* of them. Reminder: you may cite previous homework problems as references for your solutions.

Policy update: problem sets are now due by 5pm, in person or my department mailbox.

1. Verify the following statements made in lecture.
   (a) For R a ring and $f \in R$, the morphism $R \to R_f$ of rings defines a continuous map $\operatorname{Spec} R_f \to \operatorname{Spec} R$ which restricts to a homeomorphism of $\operatorname{Spec} R_f$ with $D(f) \subseteq \operatorname{Spec} R$.
   (b) For $S = \bigoplus_{n=0}^{\infty} S_n$ a graded ring and $f \in S_d$ for some $d > 0$, there is a natural homeomorphism $\operatorname{Spec} S_{f,0} \cong D_+(f)$.
2. Let R be a ring and let d be a positive integer. Compute the global sections of the structure sheaf on $\mathbb{P}^d_R$, and use the result to show that $\mathbb{P}^d_R$ is not affine.
3. A scheme X is *reduced* if for every open subset U of X, the ring $\mathcal{O}_X(U)$ has no nonzero nilpotent elements.
   (a) Prove that for R a ring, $\operatorname{Spec} R$ is reduced if and only if R has no nonzero nilpotent elements.
   (b) Prove that X is reduced if and only if for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ has no nonzero nilpotent elements.
   (c) Prove that the embedding of the category of reduced schemes into the category of all schemes has a right adjoint $X \mapsto X_{\mathrm{red}}$.
4. Let $\varphi : A \to B$ be a morphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of schemes. Prove that if φ is surjective, then f defines a homeomorphism of $\operatorname{Spec} B$ onto a closed subspace of $\operatorname{Spec} A$, and the map of sheaves $f^\sharp : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective. This statement has a converse which we will discuss in class.
5. Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a graded ring.
   (a) Prove that for any $m > 0$, the graded ring $S_0 \oplus \bigoplus_{n=m}^{\infty} S_n$ has the same Proj as does S.
   (b) Prove that for any $m > 0$, the graded ring $\bigoplus_{n=0}^{\infty} S_{mn}$ has the same Proj as does S.
6. Read the Wikipedia entry for "Veronese surface", then describe a morphism of graded rings corresponding to the inclusion $\mathbb{P}^2_R \to \mathbb{P}^5_R$ for R an arbitrary base ring.