Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 4 (due Wednesday, February 3 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let $\mathcal{F}$ be a quasicoherent sheaf on a scheme X. Prove that "$\mathcal{F}(\operatorname{Spec} R)$ is a finitely generated R-module" is a local property of an open affine subscheme $\operatorname{Spec} R$ of X, then state the conclusion that follows from the affine communication lemma. In this case, we say that $\mathcal{F}$ is *locally finitely generated*.

2. Let X be a scheme. For $x \in X$, let $\mathfrak{m}_x$ and $\kappa(x)$ denote the maximal ideal and the residue field, respectively, of the local ring $\mathcal{O}_{X,x}$. Let $\mathcal{F}$ be a quasicoherent sheaf on X which is locally finitely generated. Prove that the function $x \mapsto \dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ on X is upper semicontinuous.

3. Let $\mathcal{F}$ be a quasicoherent sheaf on a scheme X.

   (a) Prove that "$\mathcal{F}(\operatorname{Spec} R)$ is a finite R-module which is locally free (of rank n)" is a local property of an open affine subscheme $\operatorname{Spec} R$ of X. In this case, we say that $\mathcal{F}$ is *locally finite free (of rank n)*. (Warning: this need not hold if we take "locally" out of the statement of the property.)

   (b) A *vector bundle of rank* n on X is a scheme Y over X (i.e., a scheme Y plus a morphsim $Y \to X$) equipped with a section $e : X \to Y$ of the map $Y \to X$ for which we can find a covering of X by open affine subschemes U_i such that $Y \times_X U_i \cong \mathbb{A}^n_{U_i}$ with $e|_{U_i}$ corresponding to the zero section $U_i \to \mathbb{A}^n_{U_i}$. (Warning: this property does not satisfy affine communication.) Prove that in this case, the presheaf
$$U \mapsto \operatorname{Mor}_{\mathbf{Sch}_U}(U, Y \times_X U)$$
   is locally finite free of rank n (where $\mathbf{Sch}_U$ is the category of schemes over U). Optional: prove that this defines an equivalence of categories between vector bundles of rank n and quasicoherent sheaves on X which are locally free of rank n.

4. Let $\mathcal{F}$ be a locally finitely generated quasicoherent sheaf on a *reduced* scheme X such that for some nonnegative integer n, we have $\dim_{\kappa(x)} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = n$ for all $x \in X$. Prove that $\mathcal{F}$ is locally finite free of rank n. Optional: find a counterexample in case X need not be reduced.

5. (a) A morphism $f : Y \to X$ of schemes is *affine* if for every open affine subscheme $\operatorname{Spec} R$ of X, the scheme $Y \times_R \operatorname{Spec} R$ is affine. Prove that this property is local on the target and stable under base change. Hint: use a problem from a previous homework.

(b) A morphism $f : Y \to X$ of schemes is *finite* if for every open affine subscheme $\operatorname{Spec} R$ of X, the scheme $Y \times_X \operatorname{Spec} R$ is the affine scheme corresponding to a finitely generated R-algebra. Prove that this property is local on the target and stable under base change.

(c) A morphism $f : Y \to X$ of schemes is *quasi-finite* if the inverse of each point is a finite set. Prove that every finite morphism is quasi-finite, but not conversely.

6. (a) Make the following statement rigorous, and then prove it: for L/K a finite Galois extension of fields, $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is the disjoint union of the graphs of the automorphisms of L over K.

   (b) Let L/K be a finite *purely inseparable* extension of fields. Show that if $L \neq K$, then $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is not reduced. This means that the property of a morphism having reduced fibers is not stable under base change.

7. Read the proof of Proposition 7.2.9 in the Gathmann notes, then prove the following generalization. A morphism $f : X \to Y$ of schemes is *quasicompact* if Y can be covered by open affine subspaces whose inverse images are quasicompact. The morphism f is *quasiseparated* if the diagonal morphism $X \to X \times_Y X$ is quasicompact. (These properties are both local on the target, but you don't need to prove this.) Prove that if f is quasicompact and quasiseparated, then the pushforward of any quasicoherent sheaf on X is a quasicoherent sheaf on Y.