Math 203B (Algebraic Geometry), UCSD, winter 2016
Solutions for problem set 5

1. The correct formulas are

$$\begin{aligned}
F^0(M) &= \ker(\times T_1 : M \to M) \cap \ker(\times T_2 : M \to M) \\
F^1(M) &= \frac{\{(m_1, m_2) \in M \times M : T_2 m_1 = T_1 m_2\}}{(T_1 m, T_2 m) : m \in M} \\
F^2(M) &= M/(\mathrm{image}(\times T_1 : M \to M) + \mathrm{image}(\times T_2 : M \to M)) \\
F^i(M) &= 0 \qquad (i \geq 3).
\end{aligned}$$

In other words, these are the cohomology groups of the complex

$$0 \to M \stackrel{m \mapsto (T_1 m, T_2 m)}{\longrightarrow} M \times M \stackrel{(m_1, m_2) \to (T_2 m_1 - T_1 m_2)}{\longrightarrow} M \to 0.$$

The proof that these form a universal cohomological functor is similar to the one-variable case done in class.

2. For any exact sequence (in the usual orientation)

$$0 \to M \to N \to P \to 0$$

the snake lemma yields an exact sequence

$$0 \to M[f] \to N[f] \to P[f] \to M/fM \to N/fN \to P/fP \to 0,$$

where $M[f] = \ker(\times f : M \to M)$. Consequently, the derived functors are

$$\begin{aligned}
F^0(M) &= M/fM \\
F^1(M) &= M[f] \\
F^i(M) &= 0 \qquad (i \geq 2).
\end{aligned}$$

To see that these form a universal cohomological functor, let $F^0 \to F'^0$ be a morphism of functors and let F'^i be a cohomological functor. To obtain the correct morphism $F'^1(P) \to F^1(P) = P[f]$, choose an exact sequence (in the usual orientation)

$$0 \to M \to N \to P \to 0$$

with N a free module, so that the snake lemma yields $P[f] = \ker(M/fM \to N/fN)$; then use the maps $F'^0(M) \to M/fM, F'^0(N) \to N/fN$ to obtain a morphism $F'^1(P) \to P[f]$. To see that this morphism is independent of choices, it suffices to compare N with a larger free module $N \oplus N'$; this is similar to the example done in class.

3. (a) In one direction, if A is an abelian group which is an injective object, then it is divisible: for any positive integer n and any $a \in A$, the map $n\mathbb{Z} \to A$ taking n to a must factor through the injection $n\mathbb{Z} \to \mathbb{Z}$, and the image $a' \in A$ of $1 \in \mathbb{Z}$ satisfies $na' = a$. In the other direction, to see that A being divisible implies A being injective, using Zorn's lemma (or transfinite induction or your favorite other equivalent of the axiom of choice), it suffices to check the injectivity property for an injection $B \to C$ where C/B is generated by a single element c. If C/B is finite of order n, then by hypothesis we can divide the image of nc in A by n and send c there. If C/B is infinite, we can send c wherever we like (to 0, for example).

   (b) Let $\mathcal{F}$ be the sheaf in question. Let $\mathcal{G} \to \mathcal{H}$ be an injective morphism of sheaves; then the map $\mathcal{G}_x \to \mathcal{H}_x$ of stalks is injective for each $x \in X$. Let $\mathcal{G} \to \mathcal{F}$ be a morphism of sheaves. For each $x \in X$, we have a map $\mathcal{F}_x \to I_x$ and hence a map $\mathcal{G}_x \to I_x$ which factors through a map $\mathcal{H}_x \to I_x$. Now define the map $\mathcal{H} \to \mathcal{F}$ taking $s \in \mathcal{H}(U)$ to the element of $\mathcal{F}(U) = \prod_{x \in U} I_x$ whose x-component is the image of $s \in \mathcal{H}_x$ in I_x.

4. Start with a section $s \in \mathcal{H}(X)$. By definition, there exists a covering of X by some open sets $\{U_i\}_{i \in I}$ such that the restriction of s to each U_i lifts to some $s'_i \in \mathcal{G}(U_i)$; we must find a way to choose these lifts so that they agree on overlaps. If I is a two-element set, say $I = \{i, j\}$, we first pick s'_i arbitrarily. We then choose some lift $t_j \in \mathcal{G}(U_j)$, use the flasque condition to find some section $t'_j \in \mathcal{G}(U_j)$ whose restriction to $\mathcal{G}(U_i \cap U_j)$ equals the restriction of $t_j - s'_i$, then take $s'_j = t_j - t'_j$.

   To generalize to an arbitrary index set I, we use the axiom of choice to choose an isomorphism of I with some ordinal, so as to obtain a well-ordering. We may then construct the s'_i by transfinite induction. There is nothing to check at limit steps. To construct s'_i given s'_j for all $j < i$, we use the induction hypothesis to assemble a lift on $\mathcal{G}(U)$ for $U = \bigcup_{j<i} U_j$, then combine the lifts on U and U_i using the previous paragraph.

5. (a) Identify the closed points of $\mathbb{P}^{d-1}_K$ with the projectivization of the *dual* vector space V^*. For each $s \in V$, the subset U_s of $P \in X$ for which s generates $\mathcal{L}_P$ is an open subset, and the elements of V define a map $U \to \mathbb{P}^{d-1}_K$ (whose image is contained in the affine $(d-1)$-space corresponding to the complement of the kernel of $s : V^* \to K$). By hypothesis, the U_s cover all of X, so we get a well-defined map $X \to \mathbb{P}^{d-1}_K$.

   (b) With notation as in (a), note that U_s is precisely the inverse image of the complement of the hyperplane in $\mathbb{P}^{d-1}_K$ cut out by s. Consequently, if $s(P) = 0, s(Q) \neq 0$, then $P \in U_s$, $Q \notin U_s$ and so P and Q must have distinct images.

   (c) We may check the claim locally at a closed point $P \in X$. Let $Q \in \mathbb{P}^{d-1}_K$ be the image of P; the given condition implies that the map $\mathcal{O}_{\mathbb{P}^{d-1}_K, Q} \to \mathcal{O}_{X,P}$ induces a surjective morphism of cotangent spaces. Since $\mathcal{O}_{X,P}$ is a discrete valuation ring, this implies that the morphism of local rings itself is surjective.

6. It suffices to check that for each nonnegative integer k, the residue is invariant whenever f has pole order at most k. In this case, we can formally write $f = f_k T^{-k} + \cdots + f_{-1}T^{-1} + \cdots$, and then the coefficient of $T^{-1}\,dT$ in the image of $f\,dT$ under the substitution $T \mapsto a_1 T + a_2 T^2 + \cdots$ depends only on $f_{-k}, \dots, f_{-1}, a_1, \dots, a_k$. In fact, it can be written as some polynomial in these quantities with coefficients in $\mathbb{Z}$ depending only on k (not on the ring R).

   So now we must check that some specific polynomial in $f_{-k}, \dots, f_{-1}, a_1, \dots, a_k$ with integer coefficients is equal to the polynomial f_{-1}. But to check that a multivariate polynomial with integer coefficients is identically 0, it suffices to check that its evaluation at any complex numbers is zero, and this follows immediately from the Cauchy integral formula from complex analysis: the coefficient of $T^{-1}\,dT$ equals $1/(2\pi i)$ times the integral of $f\,dT$ around any simple closed curve which loops counterclockwise around 0 and is small enough not to contain any other singularities of f. Making a substitution of the form $T \mapsto a_1 T + \cdots + a_k T^k$ (there is no need to include any higher coefficients!) does not affect the looping property.