Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 5 (due Wednesday, February 10 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

From now on, you may assume without proof the theorem on the existence of universal cohomological functors for functors out of $\mathbf{Mod}_R$ or $\mathbf{Mod}_R^{\mathrm{opp}}$.

1. Let R be a ring. Give explicit formulas for the derived functors of the functor $F : \mathbf{Mod}_{R[T_1,T_2]} \to \mathbf{Mod}_R$ given by

$$F(M) = \ker(\times T_1 : M \to M) \cap \ker(\times T_2 : M \to M).$$

2. Let R be a ring and suppose $f \in R$ is not a zero-divisor. Give explicit formulas for the derived functors of the functor $F : \mathbf{Mod}_R^{\mathrm{opp}} \to \mathbf{Mod}_{R/fR}^{\mathrm{opp}}$ given by $F(M) = M/fM$.

3. An abelian group G is *divisible* if for all $n > 0$, the multiplication-by-n map on G is surjective.

   (a) Prove that the injective objects in $\mathbf{Ab}$ are precisely the divisible abelian groups. This is easy to look up, but please try to do this yourself first.

   (b) Let X be a topological space. For each $x \in X$, let I_x be a divisible abelian group. Prove that the sheaf $U \mapsto \prod_{x \in U} I_x$ is an injective object in the category of sheaves of abelian groups on X.

   I realize that both of these statements are easy to look up, but please try to prove them by hand first!

4. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of sheaves of abelian groups on a topological space X. Prove that if $\mathcal{F}$ is flasque, then

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

   is exact. Then deduce that if $\mathcal{F}, \mathcal{G}$ are flasque, then so is $\mathcal{H}$. (Hint: you might find it easier to first do the case where you are trying to lift $s \in \mathcal{H}(X)$ which lifts on each element of a *finite* cover of X, by induction on the number of opens. To get from there to the general case, you'll need some form of the axiom of choice, say Zorn's lemma or transfinite induction.)

5. Let X be a variety over an algebraically closed field K. Let $\mathcal{L}$ be a line bundle over X. Let V be a finite-dimensional subspace of $H^0(X, \mathcal{L})$ of dimension d.

   (a) Suppose that for every closed point $P \in X$, there exists $s \in V$ such that $s(P) \neq 0$. Prove that V defines a map $f : X \to \mathbb{P}_K^{d-1}$.

   (b) Suppose in addition that for every two distinct closed points $P, Q \in X$, there exists $s \in V$ such that $s(P) = 0$, $s(Q) \neq 0$. Prove that the map f is injective.

(c) Suppose in addition that X is smooth of dimension 1 and that for every closed point $P \in X$, there exists $s \in V$ such that s vanishes to exact order 1 at P. Prove that the map f is a closed immersion.

6. Let R be any ring. For f in the formal Laurent series ring $R((T))$, the *residue* of the formal differential form $f\,dT$ is defined as the coefficient of $T^{-1}\,dT$. Prove that the residue of $f\,dT$ is invariant under any substitution of the from $T \mapsto a_1 T + a_2 T^2 + \cdots$ with $a_1 \in R^\times$ and $a_2, a_3, \ldots \in R$. Hint: reduce this to a collection of polynomial identities over $\mathbb{Z}$, which can then be checked using the usual residue theorem from complex analysis.