Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 6 (due Wednesday, February 17 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Prove by a direct computation of Čech cohomology that $H^1(\mathbb{P}^2_R, \mathcal{O}(n)) = 0$ for all $n \in \mathbb{Z}$.

2. Let K be a field. Compute the Hilbert polynomials of the following schemes.

   (a) A curve of degree d in $\mathbb{P}^2_K$.

   (b) A rational normal curve in $\mathbb{P}^3_K$, that is, the Zariski closure in $\mathbb{P}^3_K$ of $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3_K$.

   (c) The Zariski closure of $\mathbb{P}^3_K$ of the union of the three coordinate axes in $\mathbb{A}^3_K$.

3. Let K be an algebraically closed field. Let $X \subseteq \mathbb{P}^d_K$ be an irreducible closed subvariety of dimension 1. Prove that X can be written as the union of two open affine subvarieties whose intersection is also affine; deduce as a corollary that for every quasicoherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for all $i > 1$. (Hint: look at the intersections of X with the complements of hyperplanes.)

4. Let $f : Y \to X$ be a morphism of schemes. Prove that the statement "$Y \times_X \operatorname{Spec}(R)$ is a union of open subschemes which are the spectra of finitely generated R-algebras" is a local property in the sense of the affine communication lemma. If this holds, we say f is *locally of finite type*. (If you only need finitely many opens each time, we say f is *of finite type*; this is quasicompact + locally of finite type.)

5. Let K be an algebraically closed field. Let $X \subseteq \mathbb{P}^d_K$ be an irreducible closed subvariety of dimension 1. Prove that there exists a finite morphism $X \to \mathbb{P}^1_K$. (Hint: project away from a point.)

6. Let K be an algebraically closed field. Show that there is a unique way to assign a *residue* to each meromorphic differential ω on $\mathbb{P}^1_K$ at each closed point P of $\mathbb{P}^1_K$ satisfying the following conditions. (A *meromorphic differential* is a section of $\Omega_{\mathbb{P}^1_K/K}$ over some nonempty open subscheme.)

   (i) For $P = 0$, the residue is computed by writing $\omega = f dT$ and taking the residue of $f\,dT$ (i.e., the coefficient of $T^{-1}\,dT$).

   (ii) If L is a linear fractional transformation, then the residue of ω at $L(P)$ is the same as the residue of $L^*(\omega)$ at P. Here L^* is the formal pullback of ω; in equations, if $L(z) = (az+b)/(cz+d)$ and $\omega = f(z)\,dz$, then

$$L^*(\omega) = f\left(\frac{az+b}{cz+d}\right) \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) dz.$$

7. Let K be an algebraically closed field. Prove the *residue theorem* for $\mathbb{P}^1_K$: for any meromorphic differential ω on $\mathbb{P}^1_K$, the sum of the residues of ω over all points of $\mathbb{P}^1_K$ (as defined in the previous exercise) is equal to 0. Hint: one possible approach is reduction to the case $K = \mathbb{C}$ by formulating the problem as a collection of polynomial identities.