Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 7 (due *Friday*, February 26 in class)

Solve the following problems, and turn in the solutions to *four* of them. Throughout this problem set, let K be an algebraically closed field.

1. In this problem, we show that if $f : X \to Y$ is a morphism of schemes, it is not always true that the image of an open affine subscheme of X is contained in an open affine subscheme of Y.

   (a) Let Z be the affine 4-space over K identified with the space of 2×2 matrices. Prove that there is an open affine subscheme X of Z whose closed points are the invertible 2×2 matrices over K.

   (b) Construct a surjective morphism $X \to \mathbb{P}^1_K$. Hint: $\mathrm{GL}_2(K)$ acts on $\mathbb{P}^1_K$ via linear fractional transformations.

2. Let C be a curve over K. Let $g : C \to \mathbb{P}^1_K$ be a finite surjective map. Prove that for any nonzero $g \in K(C)$, the divisor of g has the same degree as the divisor of $\mathrm{Norm}_{K(C)/K(\mathbb{P}^1_K)}\, g$; then deduce that this degree equals 0.

3. Let C be a curve over K.

   (a) Prove that every divisor of degree 0 on C is principal if and only if $C \cong \mathbb{P}^1_K$. (Hint: consider a divisor of the form $(P) - (Q)$, and use the resulting function to define a map $C \to \mathbb{P}^1_K$.)

   (b) Prove that every line bundle $\mathcal{L}$ of degree 0 on C is trivial if and only if $C \cong \mathbb{P}^1_K$.

4. Let $\mathcal{F}$ be a vector bundle of rank 2 over $\mathbb{P}^1_K$.

   (a) Suppose that that there exists a short exact sequence

   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

   in which $n_1 \geq n_2$. Prove that the exact sequence splits. Hint: reduce to the case $n_2 = 0$, then use the long exact sequence in cohomology.

   (b) Suppose that there exists a short exact sequence

   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

   in which $n_1 \leq n_2 - 1$. Prove that there also exists a short exact sequence

   $$0 \to \mathcal{O}(n_1 + c) \to \mathcal{F} \to \mathcal{O}(n_2 - c) \to 0$$

   for some positive integer c. Hint: this time, reduce to the case $n_1 = -1$ and remember that every line bundle on $\mathbb{P}^1_K$ of degree n is isomorphic to $\mathcal{O}(n)$. But be careful: the quotient of two vector bundles is not always a vector bundle!

5. Let $\mathcal{F}$ be a vector bundle of rank d over $\mathbb{P}^1_K$.

   (a) Prove that there exists a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F}$ of $\mathcal{F}$ by vector subbundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to $\mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$. Hint: use the fact that $\mathcal{F}(n)$ is generated by global sections for n sufficiently large.

   (b) Using (a) and the previous exercise, prove that there exists an isomorphism

   $$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d)$$

   for some $n_1, \ldots, n_d \in \mathbb{Z}$ (but not necessarily the same ones you found in (a)). Hint: note that the sum of the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ is independent of the filtration. Then use the previous exercise to raise the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for small values of i at the expense of larger values.

6. Assume K is not of characteristic 2. Let C be the Zariski closure in $\mathbb{P}^2_K$ of the zero locus of $y^2 - P(x)$ in $\mathbb{A}^2_K$, where $P(x)$ is a polynomial of degree 3 with no repeated roots.

   (a) Check that C is smooth.

   (b) Prove that the rational section dx/y of Ω is actually a global section.

   (c) Prove that the bundle Ω is trivial.

7. For C as in the previous exercise, choose a closed point $O \in C$. Prove that for any two closed points $P, Q \in C$, there exists a unique closed point $R \in C$ such that $(P) - (O) + (Q) - (O)$ and $(R) - (O)$ differ by a principal divisor.