Solve the following problems, and turn in the solutions to four of them. Note that there are no lectures during the week February 29-March 4.

Optional extra-credit problem (half credit): redo PS 5, problem 1 with the correct answer: the derived functors are the cohomology groups of the complex

\[ 0 \to M \to M \oplus M \to M \to 0 \]

where the first map takes \( m \) to \((T_1m, T_2m)\) and the second map takes \((m_1, m_2)\) to \((T_1m_2 - T_2m_1)\). For full credit, you must check not just that this gives a cohomological functor, but also that it is universal.

1. Using the cohomology of projective space (but not the Riemann-Roch theorem), prove directly that for \( C \) a smooth plane curve over an algebraically closed field \( k \), one has \( H^1(C, \Omega_{C/k}) \cong k \).

2. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Verify the Riemann-Hurwitz formula for the map \( f : \mathbb{P}^1_k \to \mathbb{P}^1_k \) given by \( x \mapsto z = x^p - x \). (Note that this map is separable.)

3. Let \( C \) be a curve over an algebraically closed field \( k \) of genus \( g \geq 2 \).
   (a) Prove that the canonical linear system is always basepoint-free, and therefore defines a morphism \( C \to \mathbb{P}^{g-1}_k \).
   (b) Prove that this morphism is a closed immersion if and only if \( C \) is not hyperelliptic.
   (c) Suppose that \( C \) is not hyperelliptic. Compute the Hilbert polynomial of \( C \) as a closed subscheme of \( \mathbb{P}^{g-1}_k \).
   Hint: use Riemann-Roch.

4. (a) Let \( C \) be a curve of genus 3 over an algebraically closed field \( k \). Prove that either \( C \) is hyperelliptic, or \( C \) is isomorphic to a smooth plane curve of degree 4.
   (b) Let \( C \) be a curve of genus 4 over an algebraically closed field \( k \). Prove that either \( C \) is hyperelliptic, or \( C \) is isomorphic to the intersection of a degree 2 surface and a degree 3 surface in \( \mathbb{P}^3_k \).

5. Let \( k \) be an algebraically closed field of characteristic \( \neq 2 \).
   (a) Let \( C \) be the plane curve \( x^4 + y^4 + z^4 = 0 \) over \( k \). Since it is smooth, we know from previous calculations that \( H^1(C, \Omega) \) is a 3-dimensional vector space over \( k \). Give an explicit formula for three linearly independent sections of \( \Omega \).
(b) Let $P(x) \in k[x]$ of degree $2g + 1$ with no repeated roots, and let $C$ be the hyperelliptic curve coming from the affine curve $y^2 = P(x)$. Give an explicit formula for $g$ linearly independent sections of $\Omega$.

In both cases, you should check that your elements are linearly independent.

6. A scheme $X$ is separated if the diagonal morphism $X \to X \times_Z X$ is a closed immersion. This is the schematic analogue of the Hausdorff condition on topological spaces.

(a) Prove that any affine scheme is separated.

(b) Let $k$ be a field, and let $X$ be the union of two copies of $\mathbb{A}^1_k$ glued along the complement of the closed point $t = 0$. Prove that $X$ is not separated.

(c) Give an example of a scheme in which the intersection of some two open affine subspaces fails to be affine. Hint: modify the example from (b).

7. (a) Prove that if $X$ is a separated scheme, then the intersection of any two open affine subspaces of $X$ is again affine. (Hint: write the intersection as a fiber product.)

(b) Let $X$ be a scheme in which any two points of $X$ are contained in some open separated subscheme. Prove that $X$ is separated.

(c) Use (b) to prove that for any ring $R$, the scheme $\mathbb{P}^n_R$ is separated.

8. Solve Hartshorne exercise IV.2.5, which proves Hurwitz’s theorem on the automorphism groups of curves over a field of characteristic 0.