Math 203B (Algebraic Geometry), UCSD, winter 2016
Problem Set 8 (due Wednesday, March 9 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them. Note that there are no lectures during the week February 29-March 4.

Optional extra-credit problem (half credit): redo PS 5, problem 1 with the correct answer: the derived functors are the cohomology groups of the complex

$$0 \to M \to M \oplus M \to M \to 0$$

where the first map takes m to $(T_1 m, T_2 m)$ and the second map takes (m_1, m_2) to $(T_1 m_2 - T_2 m_1)$. For full credit, you must check not just that this gives a cohomological functor, but also that it is universal.

1. Using the cohomology of projective space (but not the Riemann-Roch theorem), prove directly that for C a smooth plane curve over an algebraically closed field k, one has $H^1(C, \Omega_{C/k}) \cong k$.

2. Let k be an algebraically closed field of characteristic $p > 0$. Verify the Riemann-Hurwitz formula for the map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $x \mapsto z = x^p - x$. (Note that this map is separable.)

3. Let C be a curve over an algebraically closed field k of genus $g \geq 2$.

   (a) Prove that the canonical linear system is always basepoint-free, and therefore defines a morphism $C \to \mathbb{P}^{g-1}_k$.

   (b) Prove that this morphism is a closed immersion if and only if C is not hyperelliptic.

   (c) Suppose that C is not hyperelliptic. Compute the Hilbert polynomial of C as a closed subscheme of $\mathbb{P}^{g-1}$.

   Hint: use Riemann-Roch.

4. (a) Let C be a curve of genus 3 over an algebraically closed field k. Prove that either C is hyperelliptic, or C is isomorphic to a smooth plane curve of degree 4.

   (b) Let C be a curve of genus 4 over an algebraically closed field k. Prove that either C is hyperelliptic, or C is isomorphic to the intersection of a degree 2 surface and a degree 3 surface in $\mathbb{P}^3_k$.

5. Let k be an algebraically closed field of characteristic $\neq 2$.

   (a) Let C be the plane curve $x^4 + y^4 + z^4 = 0$ over k. Since it is smooth, we know from previous calculations that $H^1(C, \Omega)$ is a 3-dimensional vector space over k. Give an explicit formula for three linearly independent sections of Ω.

(b) Let $P(x) \in k[x]$ of degree $2g + 1$ with no repeated roots, and let C be the hyperelliptic curve coming from the affine curve $y^2 = P(x)$. Give an explicit formula for g linearly independent sections of Ω.

In both cases, you should check that your elements are linearly independent.

6. A scheme X is *separated* if the diagonal morphism $X \to X \times_{\mathbb{Z}} X$ is a closed immersion. This is the schematic analogue of the Hausdorff condition on topological spaces.

   (a) Prove that any affine scheme is separated.

   (b) Let k be a field, and let X be the union of two copies of $\mathbb{A}^1_k$ glued along the complement of the closed point $t = 0$. Prove that X is not separated.

   (c) Give an example of a scheme in which the intersection of some two open affine subspaces fails to be affine. Hint: modify the example from (b).

7. (a) Prove that if X is a separated scheme, then the intersection of any two open affine subspaces of X is again affine. (Hint: write the intersection as a fiber product.)

   (b) Let X be a scheme in which any two points of X are contained in some open separated subscheme. Prove that X is separated.

   (c) Use (b) to prove that for any ring R, the scheme $\mathbb{P}^n_R$ is separated.

8. Solve Hartshorne exercise IV.2.5, which proves Hurwitz's theorem on the automorphism groups of curves over a field of characteristic 0.