Math 203B: Algebraic Geometry
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The Riemann-Roch theorem

Throughout this lecture, let k be an algebraically closed field and let C be a *curve* over k, by which I will mean a smooth irreducible projective variety of dimension 1 over k (or rather, the associated scheme). Note: I'm switching the field label from capital K to lowercase k to free up the symbol K; see below.

1 The Riemann-Roch theorem

I stated (and briefly alluded to the proof of) the Riemann-Roch theorem in the last lecture.

Theorem 1 (Riemann-Roch). *For every line bundle $\mathcal{L}$ on C, there is a canonical perfect pairing*

$$H^0(C, \mathcal{L}) \times H^1(C, \Omega \otimes \mathcal{L}^{-1}) \to k.$$

In particular, the two vector spaces have the same dimension.

Let's now spin out some consequences of this statement.

Corollary 2. *The k-vector spaces $H^1(C, \mathcal{O}_C)$ and $H^0(C, \Omega)$ have the same dimension.*

This dimension is an important variant of C, called the *genus* of C. In the context of Riemann surfaces (i.e., when $k = \mathbb{C}$), one can show that $H^0(C, \Omega)$ is equal to the topological genus of the Riemann surface.

From now on, let g denote the genus of C.

Corollary 3. *For every line bundle $\mathcal{L}$ on C, we have $\chi(C, \mathcal{L}) = 1 - g + \deg(\mathcal{L})$.*

Proof. By choosing a rational section of $\mathcal{L}$, we can write $\mathcal{L} \cong \mathcal{O}(D)$ for some divisor D. Using the short exact sequence

$$0 \to \mathcal{O}_C(P) \to \mathcal{O}_C(D + P) \to j_* \mathcal{O}_C \to 0$$

for $j : P \to C$ the canonical embedding, we see that $\chi(C, \mathcal{O}_C(D + P)) = \chi(C, \mathcal{O}_C(D))$. By induction, it follows that $\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C) + \deg(\mathcal{L})$, and $\chi(C, \mathcal{O}_C) = 1 - g$ by the definition of the genus. $\square$

Using the duality from Riemann-Roch, we get the following statement, which itself is commonly called the *Riemann-Roch theorem*.

Corollary 4. *For every line bundle $\mathcal{L}$ on C, we have*

$$\dim_k H^0(C, \mathcal{L}) - \dim_k H^0(C, \Omega \otimes \mathcal{L}^{-1}) = 1 - g + \deg(\mathcal{L}).$$

You will often see this statement written in the following notation. Let K_C (or just K if C is to be understood) denote the divisor associated to a rational section of Ω, i.e., a divisor for which $\mathcal{O}(K_C) \cong \Omega$. Any such divisor is called a *canonical divisor* of C. The weird part is that the divisor is not in fact unique; only its equivalence class modulo principal divisors is unique. But rather than talking about the *canonical divisor class*, this terminology seems to have stuck.

Corollary 5. *For every divisor D on C, we have*

$$\dim_k H^0(C, \mathcal{O}(D)) - \dim_k H^0(C, \mathcal{O}(K_C - D)) = 1 - g + \deg(D).$$

Note that by substituting $K_C - D$ into this formula, we deduce that

$$\deg(K_C) = 2g - 2,$$

which happens to be the negative of the topological Euler characteristic.

Corollary 6. *For every divisor D on C, if $\deg(D) \geq 2g - 1$, then*

$$\dim_k H^0(C, \mathcal{O}(D)) = 1 - g + \deg(D).$$

Proof. If $H^0(C, \mathcal{O}(K_C - D)) > 0$, then $\deg(K_C - D)$ can be computed using a global section instead of a rational section, and therefore must be nonnegative. Consequently, if $\deg(D) \geq 2g - 1$, then $\deg(K_C - D) < 0$ and hence $H^0(C, \mathcal{O}(K_C - D)) > 0$. □

Corollary 7. *We have $g = 0$ if and only if $C \cong \mathbb{P}^1_K$.*

Proof. If $C \cong \mathbb{P}^1_K$, then $\deg(\Omega) = -2$ because the rational section dx has a double pole at infinity (by writing it as $d(y^{-1}) = -y^{-2}dy$ for $y = x^{-1}$). This implies $g = 0$.

Conversely, if $g = 0$, then choose two distinct points P, Q on C. The divisor $D = (P) - (Q)$ is of degree $0 \geq 2g - 1$, so $\dim_K H^0(C, \mathcal{O}(D)) = 1$. Any nonzero element of this space is a rational function on C with only one simple zero and one simple pole, so it gives rise to a map $C \to \mathbb{P}^1_K$ which is finite of degree 1, and hence an isomorphism. □

Corollary 8. *A smooth curve of degree d in $\mathbb{P}^2_K$ has genus $\binom{d-1}{2}$.*

Proof. For $j : C \to \mathbb{P}^2_K$ the inclusion, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_K}(-d) \to \mathcal{O}_{\mathbb{P}^2_K} \to j_* \mathcal{O}_C \to 0.$$

Taking the associated long exact sequence yields

$$0 = H^1(\mathbb{P}^2_K, \mathcal{O}_{\mathbb{P}^2_K}) \to H^1(C, \mathcal{O}_C) \to H^2(\mathbb{P}^2_K, \mathcal{O}_{\mathbb{P}^2_K}(-d)) \to H^1(\mathbb{P}^2_K, \mathcal{O}) = 0,$$

so this follows from the computation of the cohomology of projective space. □

Putting these two corollaries together, we see that a smooth curve of degree 2 (a/k/a a *conic* in $\mathbb{P}^2_K$) is always isomorphic to $\mathbb{P}^1_K$. This fact is classical: if you project from $\mathbb{P}^2_K$ away from a point *on* C, you get a well-defined isomorphism $C \cong \mathbb{P}^1_K$. By contrast, a smooth curve of degree 3 has genus 1; such a curve (once you fix a point to use as the origin of the group law) is an example of an *elliptic curve*.

2 More on curves of low genus