Math 203B: Algebraic Geometry
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Sheaf cohomology

We now discuss the main points of sheaf cohomology, omitting many proofs for now so that I can get to some applications. We may come back to some of the proofs later, depending on time and interest.

1 The category of sheaves

Fix a topological space X. Throughout this lecture, we consider only sheaves of abelian groups on X, as an abelian category. (This avoids some technical issues which I don't want to bring up yet.)

Theorem 1 (Grothendieck). *The category of sheaves of abelian groups on X has enough injectives.*

Sketch of proof. Let $\mathcal{F}$ be a sheaf. For each $x \in X$, choose an injective object I_x for which we can find an injective morphism $\mathcal{F}_x \to I_x$. Let $\mathcal{G}$ be the sheaf such that $\mathcal{G}(U) = \prod_{x \in U} I_x$; the maps $\mathcal{F}_x \to I_x$ then define an injective map $\mathcal{F} \to \mathcal{G}$. All that remains to check is that $\mathcal{G}$ is injective; for this, see the homework. $\square$

Let $\Gamma(X, \bullet)$ be the global sections functor. By the theorem, this functor admits a universal cohomological functor; those functors (also called *derived functors* of Γ) are commonly noted $H^i(X, \bullet)$. This means we have three notations for global sections: for $\mathcal{F}$ a sheaf,

$$\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F}).$$

Calculating these from the definition is generally hopeless. Fortunately, remember that we can use resolutions by arbitrary acyclic objects, not just injectives, in order to compute sheaf cohomology, and these are easier to come by. For example, we say $\mathcal{F}$ is *flasque* (or *flabby*, this being the closest English equivalent) if for all $U \subseteq V$, the map $\mathcal{F}(V) \to \mathcal{F}(U)$ is open. This is not typical for, say, quasicoherent sheaves, since $M \to M_f$ is almost never surjective; but for example the injective objects we used in the previous proof do have this property.

Lemma 2. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence and suppose $\mathcal{F}, \mathcal{G}$ are flasque.*

(a) Then $\mathcal{H}$ is also flasque.

(b) Moreover, the sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

is exact.

(c) We have $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. We leave (a) and (b) as exercises. To check (c), we induct on i. Also, note that we are free to change $\mathcal{G}$; so let's take it to be the sheaf constructed in the previous theorem. By (b) and the long exact sequence, $H^1(X,\mathcal{F})$ injects into $H^1(X,\mathcal{G}) = 0$ (since $\mathcal{G}$ is injective), so $H^1(X,\mathcal{F}) = 0$. Meanwhile, for $i > 1$, $H^i(X,\mathcal{F})$ is isomorphic to $H^{i-1}(X,\mathcal{H}) = 0$ by the induction hypothesis. $\square$

Aside: it's not hard to show that *every* injective sheaf is actually flasque, but we don't really need to know this (because we know how to make enough injectives which are also obviously flasque).

This is still not so useful in practice. Here is what we will really use.

Theorem 3. *Let $X = \operatorname{Spec} A$ be an affine scheme, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.*

Note that this immediately implies our earlier result that the global sections functor from quasicoherent sheaves to A-modules is exact. On the other hand, when A is noetherian, one can turn this around and use this exactness to prove the vanishing statement on cohomology; this is described in Hartshorne, Theorem III.3.5.

Another corollary of this theorem is the following. For $\mathcal{F}$ a sheaf of abelian groups on X and $\{U_i\}_{i\in I}$ an open cover of X labeled by an ordered index set I, define the *Čech cohomology groups* $\check{H}^i(X,\mathcal{F},\{U_i\})$ to be the cohomology groups of the complex

$$0 \to \prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j) \to \prod_{i<j<k} \mathcal{F}(U_i \cap U_j \cap U_k) \to \cdots$$

where $(s_i)_i \mapsto (s_i - s_j)_{i,j}$, $(s_{i,j})_{i,j} \mapsto (s_{j,k} - s_{i,k} + s_{i,j})_{i,j,k})$, and in general

$$(s_{i_1,\dots,i_m})_{i_1,\dots,i_m} \mapsto \left(\sum_{k=0}^{m} (-1)^k s_{i_0,\dots,\widehat{i_k},\dots,i_m} \right)_{i_0,\dots,i_m}$$

(where the hat means omit that index). These are not guaranteed to match the $H^i(X,\mathcal{F})$ in general, but...

Lemma 4. *If $H^j(U_{i_1} \cap \cdots \cap U_{i_m}, \mathcal{F}) = 0$ for all $j > 0$ and all $i_1 < \cdots < i_m$, then there is a canonical isomorphism $H^j(X,\mathcal{F}) \cong \check{H}^j(X,\mathcal{F},\{U_i\})$.*

Proof. This follows from the existence of a certain *spectral sequence.* See Stacks Project, tag 01ET. (You might try looking at the case of a two-set cover by hand, in order to get the basic idea.) $\square$

Corollary 5. *If $\mathcal{F}$ is injective or flasque, then $\check{H}^j(X,\mathcal{F},\{U_i\}) = 0$ for all $j > 0$.*

Corollary 6. *Let $X = \operatorname{Spec} A$ be an affine scheme, let $f_1,\dots,f_n \in A$ generate the unit ideal, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then $\check{H}^j(X,\mathcal{F},\{D(f_i)\}_{i=1}^n) = 0$ for all $j > 0$.*

But we already know how to prove this directly (see past homework)! In fact, with some effort, one can turn around and use this as the starting point to prove that quasicoherent sheaves on affine schemes are acyclic; see Lemma 8.

2 Some additional arguments

These will not be presented in class, but are here for your edification.

For $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X, we say that $\mathcal{F}$ is *$\mathcal{U}$-acyclic* if $\check{H}^j(X, \mathcal{F}, \mathcal{U}) = 0$ for all $j > 0$. We'll say that $\mathcal{F}$ is *strongly Čech-acyclic* if there exist a cofinal set of open coverings $\mathcal{U}$ (i.e., every open covering can be refined to one of the given ones) for each of which $\mathcal{F}$ is $\mathcal{U}$-acyclic. This implies that $\mathcal{F}$ is *Čech-acyclic*, meaning that $\check{H}^j(X, \mathcal{F})$ vanishes for all $j > 0$, where $\check{H}^j(X, \mathcal{F})$ is defined as the direct limit of $\check{H}^j(X, \mathcal{F}, \mathcal{U})$ over all coverings $\mathcal{U}$.

In general, we can't say that $H^j(X, \mathcal{F}) = \check{H}^j(X, \mathcal{F})$ for all j; but this is definitely true for $j = 0$, because in fact $H^0(X, \mathcal{F}) = \check{H}^0(X, \mathcal{F}, \mathcal{U})$ for each $\mathcal{U}$.

Lemma 7. *If $\check{H}^1(X, \mathcal{F}) = 0$, then for any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ the sequence*

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

is exact.

Proof. As usual, it suffices to check that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective. Given $s \in \mathcal{H}(X)$, there exists an open covering $\mathcal{U}$ of X such that for each $i \in I$, s lifts to a section $t_i \in \mathcal{G}(U_i)$. The differences $t_i - t_j$ then define an element of $\check{H}^1(X, \mathcal{F}, \mathcal{U})$; by hypothesis, we can replace $\mathcal{U}$ by a finer cover so that the element of $\check{H}^1(X, \mathcal{F}, \mathcal{U})$ vanishes. That is, there exist elements $u_i \in \mathcal{F}(U_i)$ with $u_i - u_j = t_i - t_j$; the elements $t_i - u_i$ then glue to a section in $\mathcal{G}(X)$ lifting s. □

Lemma 8. *If $\mathcal{F}|_U$ is Čech-acyclic for each open set U in some basis of the topology of X, then $\mathcal{F}$ is acyclic.*

Proof. We will prove that $H^i(X, \mathcal{F}) = 0$ for $i = 1, \dots, m$ by induction on m, with empty base case $m = 0$. Suppose the claim is given for some m. Given $\mathcal{F}$, form the usual exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

where $\mathcal{G}$ is injective and flasque. By Lemma 7, taking sections over any open subset in a basis of X gives an exact sequence. By the snake lemma, it follows that $\mathcal{H}$ is flasque and hence Čech-acyclic by Corollary 5.

To prove the claim for $m = 1$, note that by Lemma 7, $H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$ is surjective. Therefore, in the exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) = 0,$$

the connecting homomorphism $H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$ is zero. Therefore $H^1(X, \mathcal{F})$ is trapped between a zero map and a zero object, so it must be zero.

To prove the claim for $m + 1$ given the claim for m, note that

$$0 = H^m(X, \mathcal{G}) \to H^m(X, \mathcal{H}) \to H^{m+1}(X, \mathcal{F}) \to H^{m+1}(X, \mathcal{G}) = 0$$

is exact and $H^m(X, \mathcal{H}) = 0$ by the induction hypothesis, so $H^{m+1}(X, \mathcal{F}) = 0$ also. □