Math 203B: Algebraic Geometry
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Motivation for schemes

In Math 203A, algebraic geometry was presented in the language of varieties. This formulation involves two key ingredients. One of these ingredients is Hilbert's *Nullstellensatz*, which identifies the points of an affine space over an algebraically closed field with the maximal ideals of a polynomial ring; this makes it possible to translate geometric statements about points into statements of algebra. The other ingredient is Cartan's definition of *sheaves* (adopted in algebraic geometry by Serre), which can be used to assemble spaces together out of affine pieces (e.g., projective varieties) in a fashion closely analogous with the construction of various classes of manifolds.

While the language of varieties is sufficient to articulate many statements of algebraic geometry, we are fortunate to have a much more flexible and powerful framework available to replace it: Grothendieck's language of *schemes*. In the world of schemes, one glues together objects built not out of polynomial rings over a field and quotients thereof, but entirely arbitrary rings![1] This incorporates Zariski's insight that for a general ring, it is the set of *prime* ideals, rather than maximal ideals, that can be provided with a meaningful topology.

In this lecture, rather than starting directly with the definition, I would instead like to give a summary of a few key ideas we will be encountering in this course (and Math 203C).

- **Not all objects are irreducible.**

  In classical algebraic geometry, only irreducible spaces were allowed to be called varieties; for instance, the union of two distinct lines in the affine or projective plane was excluded. From the point of view of schemes, there is no reason at all to exclude reducible spaces.

- **Not all objects are reduced.**

  Under the Nullstellensatz, algebraic subsets of an affine space correspond to radical ideals of the polynomial ring. But what about nonradical ideals? In the world of schemes, they also correspond to geometric objects. This makes sense if one thinks about geometric families, e.g., conics in the plane. If one considers a family of plane conics over $\mathbb{C}$ cut out by
  $$x^2 + y^2 + tz^2$$
  where t is a varying parameter, the degenerate case $t = 0$ corresponds to a pair of distinct lines. But in the family
  $$x^2 + t(y^2 + z^2),$$
  the degenerate case $t = 0$ corresponds to the single line $t = 0$. In order for this degeneration to behave well (e.g., to preserve the Hilbert polynomial), we must treat it as *two* copies of the line, which we can naturally do in the context of schemes.

[1] Once and for all, throughout Math 203, all rings will be commutative and with unit.

- **Not all spaces are Hausdorff (separated).**

  In classical algebraic geometry, varieties are usually required to be Hausdorff. So for example, you cannot glue together two copies of the affine line along the complement of 0. In the world of schemes, you can do this. (Originally this was excluded from the definition of a scheme also, but this was later judged to be a mistake and the terminology was changed.)

- **Not all fields are algebraically closed.**

  In many applications of algebraic geometry, the natural choice of a base field is not algebraically closed. For instance, algebraic geometry over the real numbers has applications in combinatorial geometry, statistics, robotics, computer vision, and more. Algebraic geometry over finite fields has applications in combinatorial geometry, coding theory, cryptography, and more. And of course, algebraic geometry over number fields lies at the heart of number theory. While one can adapt the statement of the Nullstellensatz to handle fields which are not algebraically closed, this is a more complicated solution than switching to schemes (a bit like patching the geocentric model of planetary motion with epicycles rather than switching to the heliocentric model).

- **Think categorically.**

  In the 20th century, it became clear to mathematicians that structured mathematical objects (e.g., algebraic structures like groups and rings, topological structures like topological spaces and manifold, etc.) are best handled by treating them as members of some sort of over-structure that keeps track both of the objects themselves, and of the structure-preserving maps between them. The appropriate abstraction for this is the notion of a *category*; the members of a category are called *objects*, and the structure-preserving maps between objects are called *morphisms*. (E.g., think of rings and ring homomorphisms, or topological spaces and continuous maps.)

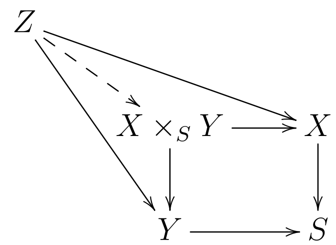
  Thinking categorically has the advantage of making it easier to guess the correct analogues of standard constructions when working in an unfamiliar environment: typically, the recipe for building the correct analogous object is less important than the universal properties that it satisfies. For schemes, one important example of this is products: the product of two schemes X and Y should be a scheme $X \times Y$ equipped with morphisms $X \times Y \to X, X \times Y \to Y$ with the property that for every scheme Z and every pair of morphisms $Z \to X, Z \to Y$, there should be a unique morphism $Z \to X \times Y$ such that the compositions $Z \to X \times Y \to X$, $Z \to X \times Y \to Y$ agree with the original ones. In particular, $X \times Y$ is "unique up to unique isomorphism": given any other candidate (again equipped with maps to X and Y) satisfying the same condition, each one is forced to admit a unique map to the other. (Thought exercise: convince yourself that in the category of sets, this characterizes the usual Cartesian product.)

- **Work relatively.**

Continuing in the categorical vein, in the theory of schemes, the familiar adjectives of algebraic geometry (e.g., projective, smooth) typically apply not to individual objects, but rather to morphisms. How this relates to the theory of varieties is that a variety defined over a field K corresponds to a scheme X plus a morphism from that scheme to (the scheme corresponding to) K itself, and properties of the original variety will correspond to properties of that so-called *structure morphism*.

This becomes less counterintuitive if you think about field automorphisms. For instance, if you define a variety in terms of some equations over $\mathbb{C}$, you can apply complex conjugation to all of the coefficients to get a new variety. The corresponding schemes are abstractly isomorphic, but we can keep track of the difference by remembering the structure morphisms. If we take more exotic algebraic automorphisms over $\mathbb{C}$, we can put this point in starker relief: applying an algebraic automorphism can actually change the homotopy type of the associated topological space! The point is that in order to pass from the scheme to the topological space, one needs to use the structure morphism in an essential way.

Remembering the structure morphism also helps us understand products: the construction we gave above is not quite the right analogue of the product of varieties. The correct analogue, accounting for structure morphisms, is a *fiber product*: given three schemes X, Y, S and maps $X \to S, Y \to S$, there exists a scheme $X \times_S Y$ equipped with morphisms $X \times_S Y \to X$, $X \times_S Y \to Y$ such that the compositions $X \times_S Y \to X \to S$, $X \times_S Y \to Y \to S$ coincide, and given any other pair of morphisms $Z \to X, Z \to Y$ such that the compositions $Z \to X \to S$, $Z \to Y \to S$ coincide, there is a unique morphism $Z \to X \times_S Y$ such that the compositions $Z \to X \times_S Y \to X$, $Z \to X \times_S Y \to Y$ agree with the given ones. To avoid the pain of having to write out sentences like that repeatedly, we typically represent them in a graphical form:



with the interpretation that following any two paths between two points in the diagram should lead to sequences of morphisms with the same composition, i.e., the diagram *commutes*. (In the category of sets, the fiber product corresponds to taking the subset of the cartesian product $X \times Y$ consisting of those pairs (x, y) for which x and y have the same image in S.)

- **The base may vary.**

  As we just discussed, in the theory of schemes, geometric properties are mostly encoded as properties of morphisms rather than properties of objects. So far we have been thinking of cases where the target of this morphism is a field; however, that target can perfectly well itself be another, more interesting scheme. For example, if it is itself a variety over a field, then we are looking at a *family* of geometric objects, such as the families of conics I considered earlier (which live inside projective 2-space over an affine line). This is made extremely useful by one of the distinguishing features of algebraic geometry: often the geometric objects of a given type occur in *universal families* described by algebraic invariants (e.g., the j-invariant in the theory of elliptic curves).

One way to interpret the fiber product we discussed earlier is: if $X \to S$ is a family of objects and $Y \to S$ is any other morphism, then $X \times_S Y \to Y$ corresponds to the "pullback family" in which the fiber over a point $y \in Y$ is a copy of the fiber of the point of S to which y maps. That is, forming the fiber product amounts to *changing the base* of the original morphism; we will mostly be interested in properties of morphisms that are invariant under such a base change.

A more exotic way to exploit the relative nature of properties of morphisms is to work not over a field, but over a ring. For example, if one starts with a morphism to the ring of integers $\mathbb{Z}$, then for any prime number p, the reduction map $\mathbb{Z} \to \mathbb{F}_p$ will correspond to a morphism of schemes in the other direction (just as for varieties, this correspondence is contravariant). Given a morphism to $\mathbb{Z}$, we can then change base to $\mathbb{F}_p$; this is an algebro-geometric form of "reduction modulo p" which is impossible to articulate in the language of varieties. Number theorists use this all the time, but it occurs also in other aspects of algebraic geometry: there are statements which make no apparent reference to positive characteristic for which the easiest proof (or sometimes the only known proof) involves reduction modulo p (e.g., Mori's bend-and-break lemma in the minimal model program).

- **Not all objects are noetherian.**

  By contrast with quotients of polynomial algebras over a ring, arbitrary rings are not noetherian. Consequently, when discussing schemes, we will at certain points be forced to choose between imposing extra noetherian conditions or making weaker statements than would otherwise be possible. However, we will get quite far before being forced into this corner! In fact, it may be clarifying to see how much can be done without using a noetherian hypothesis as a crutch.

  Moreover, rings which are not noetherian do occasionally play a role in interesting algebraic geometry. My favorite example is in characteristic p, where Galois theory works best for fields which are *perfect* (i.e., the p-th power map is not only an endomorphism, but an automorphism). Similarly, certain statements about rings behave between when the rings are perfect in the same sense, but perfectness is almost entirely incompatible with noetherianness; for instance, a quotient of a polynomial ring over a field cannot be perfect unless it is itself a direct sum of fields.