Math 203B: Algebraic Geometry
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Affine communication (updated 3 Feb 2020)

The following lemma will be used repeatedly in the study of properties of schemes and properties of morphisms of schemes. The presentation, and the name *affine communication lemma* for this argument, are taken from Ravi Vakil's Math 216 lecture notes except for the reduction to the case $n = 2$; this was suggested by Ofer Gabber in private communictation.

For X a scheme, a *local property* is a property $\mathcal{P}$ of open affine subschemes of X satisfying the following axioms. Here $\mathrm{Spec}(R)$ denotes an arbitrary open affine subscheme of X.

(i) If $\mathrm{Spec}(R)$ satisfies $\mathcal{P}$, then so does $\mathrm{Spec}(R_f)$ for all $f \in R$.

(ii) If there exist $f_1, \dots, f_n \in R$ generating the unit ideal such that $\mathrm{Spec}(R_{f_i})$ satisfies $\mathcal{P}$ for $i = 1, \dots, n$, then $\mathrm{Spec}(R)$ satisfies $\mathcal{P}$.

For example, the property "R has no nonzero nilpotent elements" is local (see PS2). For another example, the property "R is noetherian" is local (see Hartshorne Proposition II.3.2).

Lemma 1 (Affine communication lemma)**.** *Let X be a scheme and let $\mathcal{P}$ be a local property of open affine subschemes of X. If X is covered by open affine subschemes satisfying $\mathcal{P}$, then* every *open affine subscheme of X satisfies $\mathcal{P}$.*

Before proving this lemma, let us explain how it will be used in the theory. We will use it in several different ways.

- *Properties of schemes*: let $\mathcal{P}$ be a property of affine schemes satisfying axioms (i) and (ii). Then we may formally extend $\mathcal{P}$ to a property of arbitrary schemes by declaring that X satisfies $\mathcal{P}$ if X is covered by open affine subschemes satisfying $\mathcal{P}$. It will then follow from the lemma that every open affine subscheme of X satisfies $\mathcal{P}$. We will say that any such property is a *local property* of schemes (e.g., reduced).

- *Properties of sheaves*: let $\mathcal{P}$ be a property of sheaves on affine schemes. Suppose that for each scheme X and each sheaf $\mathcal{F}$ on X, "the restriction of $\mathcal{F}$ to $\mathrm{Spec}(R)$ satisfies $\mathcal{P}$" is a local property. Then we may formally extend $\mathcal{P}$ to a property of sheaves on arbitrary schemes. We will say that any such property is a *local property* of sheaves of schemes (e.g., quasicoherent, finitely generated, locally free).

- *Properties of morphisms, part 1*: let $\mathcal{P}$ be a property of morphisms from an arbitrary scheme to an affine scheme. Suppose that for each morphism $f : Y \to X$ of schemes, "the restriction of f to $\mathrm{Spec}(R)$ satisfies $\mathcal{P}$" is a local property. Then we may formally extend $\mathcal{P}$ to a property of sheaves on arbitrary schemes. We will say that any such property is *local on the base* or *local on the target* (e.g., open immersion, closed immersion, finite, separated, quasicompact, quasiseparated). Note that stability under base change is a separate issue; we'll come back to that.

- *Properties of morphisms, part 2*: let $\mathcal{P}$ be a property of morphisms from an affine scheme to an arbitrary scheme. Suppose that for each morphism $f : X \to Y$ of schemes, "the restriction of f to $\mathrm{Spec}(R)$ satisfies $\mathcal{P}$" is a local property. Then we may formally extend $\mathcal{P}$ to a property of sheaves on arbitrary schemes. We will say that any such property is *local on the source.*

- *A hybrid*: let $\mathcal{P}$ be a property of a morphism $f : X \to Y$ together with a sheaf $\mathcal{F}$ on X, et cetera. This is getting ridiculous, but there is one important property (flatness) which is defined in this context.

Now to prove the lemma.

Proof of Lemma 2. By assumption, we can cover X with open affine subschemes $\{\mathrm{Spec}(S_i)_{i\in I}\}$ satisfying $\mathcal{P}$. Recall that the distinguished open affine subschemes of $\mathrm{Spec}(S_i)$ form a topological basis of that space; by this observation plus axiom (i), X admits a topological basis consisting of open affine subschemes satisfying $\mathcal{P}$.

Let $\mathrm{Spec}(R)$ be an arbitrary open affine subscheme of X. By the previous paragraph, $\mathrm{Spec}(R)$ can be covered by open affine subschemes $\mathrm{Spec}(S_i)$ satisfying $\mathcal{P}$, but these need not be distinguished. However, the distinguished open affine subschemes of $\mathrm{Spec}(R)$ form a basis, so we may choose elements $f_j \in R$ such that the schemes $\mathrm{Spec}(R_{f_j})$ cover $\mathrm{Spec}(R)$ and each $\mathrm{Spec}(R_{f_j})$ is contained in some $\mathrm{Spec}(S_i)$. As usual, the $\mathrm{Spec}(R_{f_j})$ cover $\mathrm{Spec}(R)$ if and only if the f_j generate the unit ideal in R, so we need only keep the finitely many of them used in some specific representation of 1; that is, we may take the f_j to be $f_1, \dots, f_n$ for some n.

The key point now is that the inclusion $\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)$ allows us to view f_j as an element of S_i. As open subschemes of X, we then have

$$\begin{aligned}\mathrm{Spec}((S_i)_{f_j}) &= \{\mathfrak{p} \in \mathrm{Spec}(S_i) : f_j \notin \mathfrak{p}\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}(S_i) : f_j \notin \mathfrak{p}S_{i,\mathfrak{p}}\}\end{aligned}$$

(if $\mathfrak{p} \in \mathrm{Spec}(S_i)$ maps to $\mathfrak{q} \in \mathrm{Spec}(R)$, then $S_{i,\mathfrak{p}} = \mathcal{O}_{\mathrm{Spec}(S_i),\mathfrak{p}} = \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{q}} = R_{\mathfrak{q}}$)

$$\begin{aligned}&= \mathrm{Spec}(S_i) \cap \{\mathfrak{q} \in \mathrm{Spec}(R) : f_j \notin \mathfrak{q}R_{\mathfrak{q}}\}\\ &= \mathrm{Spec}(S_i) \cap \{\mathfrak{q} \in \mathrm{Spec}(R) : f_j \notin \mathfrak{q}\}\\ &= \mathrm{Spec}(S_i) \cap \mathrm{Spec}(R_{f_j})\\ &= \mathrm{Spec}(R_{f_j}),\end{aligned}$$

so we may propagate $\mathcal{P}$ from $\mathrm{Spec}(S_i)$ to $\mathrm{Spec}((S_i)_{f_j}) = \mathrm{Spec}(R_{f_j})$ using (i) and from $\mathrm{Spec}(R_{f_1}), \dots, \mathrm{Spec}(R_{f_n})$ to $\mathrm{Spec}(R)$ using (ii). □

The following refinement is sometimes useful, for instance for the study of quasicoherent sheaves.

Lemma 2 (Strong affine communication lemma)**.** *Let X be a scheme and let $\mathcal{P}$ be a property of open affine subschemes of X satisfying (i) and the restricted version of (ii) in which $n = 2$ (for arbitrary R). Then $\mathcal{P}$ is a local property, so affine communication applies.*

Proof. We prove that property (ii) holds for arbitrary n by induction, the case $n = 1$ being trivial and $n = 2$ being given. Given $f_1, \dots, f_n \in R$ generating the unit ideal, choose $a_1, \dots, a_n \in R$ satisfying $a_1 f_1 + \cdots + a_n f_n = 1$ and put

$$g = a_1 f_1 + \cdots + a_{n-1} f_{n-1};$$

then g and f_n also generate the unit ideal because $g + a_n f_n = 1$. If $\mathrm{Spec}(R_{f_i})$ satisfies $\mathcal{P}$ for all i, then so does $\mathrm{Spec}((R_{f_i})_g) = \mathrm{Spec}(R_{f_i g}) = \mathrm{Spec}((R_g)_{f_i})$ for all i by (i). In R_g, the elements $f_1, \dots, f_{n-1}$ generate the unit ideal, so we may deduce that $\mathrm{Spec}(R_g)$ satisfies $\mathcal{P}$. Since $\mathrm{Spec}(R_g)$ and $\mathrm{Spec}(R_{f_n})$ satisfy $\mathcal{P}$, so does $\mathrm{Spec}(R)$. □

An important application is to stability under base change.

Lemma 3. *Let $\mathcal{P}$ be a property of morphisms of schemes which is local on the target. Suppose moreover that for every morphism $f : Y \to \mathrm{Spec}(R)$ of schemes satisfying $\mathcal{P}$ and every morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of affine schemes, the morphism $Y \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \to \mathrm{Spec}(S)$ satisfies $\mathcal{P}$. Then $\mathcal{P}$ is* stable under base change*: for every morphism $f : Y \to X$ satisfying $\mathcal{P}$ and every morphism $g : Z \to X$, the morphism $Y \times_X Z \to Z$ satisfies $\mathcal{P}$.*

Proof. Straightforward. □