Math 203B: Algebraic Geometry
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Projective and proper morphisms

In this lecture, we will need to pass freely between closed subsets and closed subschemes. In general the same closed subset corresponds to multiple closed subschemes, but not if we consider only reduced schemes.

Lemma 1. *Let X be a scheme.*

(a) Every closed subset S of X is the image of a unique closed immersion $i : S_{\mathrm{red}} \to X$ with S reduced.

(b) In (a), if $f : Y \to X$ is a morphism of schemes with Y reduced and $f(Y) \subseteq S$, then f factors through i.

(c) There is a unique (up to unique isomorphism) closed immersion $f_X : X_{\mathrm{red}} \to X$ with X_{red} reduced and $f(X_{\mathrm{red}}) = X$.

(d) Any morphism $Y \to X$ of schemes with Y reduced factors uniquely through X_{red}.

(e) The assignment $X \mapsto X_{\mathrm{red}}$ is functorial, and for any morphism $g : Y \to X$ the diagram

$$\begin{tikzcd} Y_{\mathrm{red}} \arrow[r, "f_Y"] \arrow[d, "g_{\mathrm{red}}"] & Y \arrow[d, "g"] \\ X_{\mathrm{red}} \arrow[r, "f_X"] & X \end{tikzcd}$$

commutes.

Proof. Everything reduces to the case where $X = \operatorname{Spec} R$ is affine. To prove (a), write S as $V(I)$ for some ideal I of R and take $S_{\mathrm{red}} = \operatorname{Spec}(R/\sqrt{I})$. To prove (b), we may assume that Y is also affine, namely $Y = \operatorname{Spec}(T)$ with T reduced. The ideal $\sqrt{I}$ is the intersection of the prime ideals $\mathfrak{p}$ containing I; by hypothesis, the morphism $R \to T$ induced by f has the property that every element of $\sqrt{I}$ maps to an element contained in all of the prime ideals of T, which must be zero because T is reduced. Now note that (c) is the special case $Y = X$ of (a); (d) is the special case $Y = X$ of (b); and (e) follows from (d). $\square$

We will also need some facts about minimal prime ideals.

Lemma 2. *For any ring R and any prime ideal $\mathfrak{p}$ of R, there exists a minimal prime ideal of R contained in $\mathfrak{p}$.*

Proof. Because the intersection of a descending chain of prime ideals is a prime ideal, this follows from Zorn's lemma. $\square$

Lemma 3. *Let R be a reduced ring and let $\mathfrak{p}$ be a minimal prime ideal of R. Then $R_{\mathfrak{p}}$ is a field.*

Proof. Since R is reduced, $R_{\mathfrak{p}}$ is reduced. Since $\mathfrak{p}$ is minimal, $R_{\mathfrak{p}}$ has no prime ideals other than its maximal ideal. The intersection of the prime ideals of $R_{\mathfrak{p}}$ is on one hand equal to 0 (because R is reduced) and equal to the maximal ideal of $R_{\mathfrak{p}}$ (because this is the only prime ideal in the intersection). Hence the maximal ideal of $R_{\mathfrak{p}}$ is 0, so $R_{\mathfrak{p}}$ is a field. □

The definition of properness involves checking that various maps of schemes are *closed maps* of topological spaces (i.e., the image of any closed subset is closed). Checking this condition is facilitated by the following lemma, which in cases of interest reduces closedness to a weaker property.

Let X be a scheme. For $x, y \in X$, we say that y is a *specialization* of X if y belongs to the closure of $\{x\}$. A subset S of X is *stable under specialization* if for all $x \in S$ and all $y \in X$, if y is a specialization of x then $y \in S$. Clearly a closed set is stable under specialization, but not conversely: if $X = \operatorname{Spec} k[t]$, then every point is a specialization of the generic point, but no two other points are specializations of each other, so any infinite set of closed points is stable under specialization but not closed.

Lemma 4. *Let $f : Y \to X$ be a quasicompact morphism of schemes and suppose that $f(Y)$ is stable under specialization. Then $f(Y)$ is closed.*

Proof (from Hartshorne, Lemma II.4.5). We start with some reduction steps. Using Lemma 1, we may assume that X and Y are reduced and that f has dense image (by replacing X with the closure of $f(Y)$). Under these conditions, we must check that any given $x \in X$ belongs to $f(Y)$; for this, we may replace X with an open affine neighborhood of x. That is, we may assume that $X = \operatorname{Spec}(R)$ is affine (and still reduced).

Since f is quasicompact and X is now affine, Y is a finite union of finitely many open affine subschemes $Y_1, \dots, Y_n$. Since $f(Y) = f(Y_1) \cup \cdots \cup f(Y_n)$ is dense, there must exist an index i for which x belongs to the closure of $f(Y_i)$. (**Warning**: we cannot ensure that each $f(Y_i)$ is itself stable under specialization. That is, we cannot formally reduce the lemma to the case of an affine morphism.) By Lemma 1, the closure of $f(Y_i)$ can be viewed as the image of a closed immersion $X_i \to X$ with X_i reduced through $f : Y_i \to X$ factors. If we put $X_i = \operatorname{Spec}(R_i)$ and $Y_i = \operatorname{Spec}(S_i)$, we now have a map $\operatorname{Spec}(S_i) \to \operatorname{Spec}(R_i)$ with dense image, and moreover $x \in \operatorname{Spec}(R_i)$.

The point x corresponds to a prime ideal $\mathfrak{p}$ of R_i. Apply Lemma 2 to construct a minimal prime ideal $\mathfrak{p}'$ of R_i contained in $\mathfrak{p}$, and let x' be the corresponding point of X_i. Note that $R_{i,\mathfrak{p}'}$ is a field by Lemma 3. Also, x is a specialization of x', so to finish the proof it will suffice to check that x' belongs to the image of $Y_i \to X_i$ (and hence to the image of f).

Since $\operatorname{Spec}(S_i) \to \operatorname{Spec}(R_i)$ has dense image, $R_i \to S_i$ must be injective: any element of R_i mapping to zero in S_i gives a section of the structure sheaf on $\operatorname{Spec}(R_i)$ with zero stalks at a dense subset, and hence everywhere. Since localization is flat, $R_{i,\mathfrak{p}'} \to S_i \otimes_{R_i} R_{i,\mathfrak{p}'}$ is also injective, so in particular $S_i \otimes_{R_i} R_{i,\mathfrak{p}'}$ is a nonzero ring. It thus admits at least one prime ideal, which must contract to zero in the field $R_{i,\mathfrak{p}'}$ and thus to the ideal $\mathfrak{p}$ in R_i. This proves the stated claim. □

A morphism of schemes $f : Y \to X$ is *universally closed* if for any morphism $Z \to X$, the map $Y \times_Z X \to Z$ is a closed map of topological spaces (the image of any closed subset is closed). Note that it suffices to check this condition for Z affine and reduced (by Lemma 1). A morphism of schemes is *proper* if it is separated, of finite type, and universally closed.

For example, any closed immersion is proper: it is affine (hence separated), of finite type (trivially), and universally closed (it's enough to check closed, which is because a closed immersion is a homeomorphism to a closed subset of the target).

The properness condition is meant to capture the idea of a *compact* topological space; this suggests that projective spaces should be proper over their bases. The fact that this is correct is a validation of the definition of properness!

Theorem 5. *The map* $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ *is proper.*

The usual proof of this result uses the *valuative criterion for properness* (see, e.g., Hartshorne, Theorem II.4.7). I will give a different (but closely related) proof, in which the case $n = 1$ is treated directly, and the general case is reduced to the $n = 1$ case. (The treatment of the $n = 1$ case amounts to the construction of valuation rings as in Atiyah-Macdonald, Lemma 5.20.)

Lemma 6. *Let R be an integral domain and let L be a field containing* $\operatorname{Frac}(R)$. *For $t \in L$ nonzero, let $R[t], R[t^{-1}]$ denote the R-subalgebras of L generated by t, t^{-1}. Then the natural map*

$$\operatorname{Spec}(R[t]) \cup \operatorname{Spec}(R[t^{-1}]) \to \operatorname{Spec}(R)$$

is surjective.

Note that the same statement formally holds with $t = 0$ if we declare $R[t^{-1}]$ to be the zero ring, or with $t = \infty$ if we declare $R[t]$ to be the zero ring.

Proof. Suppose by way of contradiction that $x \in \operatorname{Spec}(R)$ is not in the image. Let $\mathfrak{p}$ be the prime ideal corresponding to x; then $\mathfrak{p}R_{\mathfrak{p}}[t] = R_{\mathfrak{p}}[t]$ and $\mathfrak{p}R_{\mathfrak{p}}[t^{-1}] = R_{\mathfrak{p}}[t^{-1}]$. This means that for some $m, n \geq 0$, there exist $u_0, \dots, u_m, v_0, \dots, v_n \in \mathfrak{p}R_{\mathfrak{p}}$ such that

$$1 = u_0 + u_1 t + \cdots + u_m t^m = v_0 + v_1 t^{-1} + \cdots + v_n t^{-n}.$$

We are free to choose m, n as small as possible. Suppose without loss of generality that $m \geq n$. Since $v_0 \in \mathfrak{p}R_{\mathfrak{p}}$, $1 - v_0$ is a unit in $R_{\mathfrak{p}}$, so we have

$$t^n = (1 - v_0)^{-1} v_1 t^{n-1} + \cdots + (1 - v_0)^{-1} v_n$$

But then I can rewrite the relation $1 = u_0 + \cdots + u_m t^m$ replacing t^m with $(1 - v_0)^{-1} v_1 t^{m-1} + \cdots + (1 - v_0)^{-1} v_n t^{m-n}$, reducing the value of m. This contradiction yields the desired result. $\square$

Although it is not logically necessary to do this case first, it may be helpful conceptually to see how Lemma 6 applies to the case $n = 1$ of Theorem 5.

Lemma 7. *Let R be an integral domain and put $K =$* $\operatorname{Frac}(R)$. *Let Z be a closed subset of $\mathbb{P}^1_R$ meeting $\mathbb{P}^1_K$. Then $Z \to \operatorname{Spec}(R)$ is surjective.*

The idea is to get surjectivity by trying to construct a section of the map $\mathbb{P}^1_R \to \operatorname{Spec}(R)$ landing inside Z. This may not quite work over R, but it will work over a slightly larger ring.

Proof. Choose $z \in Z \cap \mathbb{P}^1_K$. If z is the generic point of $\mathbb{P}^1_K$, then $Z = \mathbb{P}^1_R$ and we are done. Otherwise, z is a closed point of $\mathbb{P}^1_K$ whose residue field L is a finite extension of K. Write $z = [z_0 : z_1]$ in homogeneous coordinates with $z_0, z_1 \in L$. Put $t = z_1/z_0 \in L \cup \{\infty\}$.

If $t \neq \infty$, then t defines a section $s : \operatorname{Spec}(R[t]) \to \mathbb{P}^1_{R[t]}$ of the projection map, under which the generic point of $\operatorname{Spec}(R[t])$ maps to a point z' of $\mathbb{P}^1_L$ projecting to z in $\mathbb{P}^1_K$. In particular, the closure of z' in $\mathbb{P}^1_{R[t]}$ is the whole image of s, so the projection of this image back to $\mathbb{P}^1_R$ must be contained in Z. Therefore, any point in the image of $\operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$ is in the image of Z.

Similarly, if $t \neq 0$, then any point in the image of $\operatorname{Spec}(R[t^{-1}]) \to \operatorname{Spec}(R)$ is in the image of Z. But by Lemma 6, this covers all of $\operatorname{Spec}(R)$. $\square$

In case $z_0, z_1 \in R$, the rings $\operatorname{Spec}(R[t])$ and $\operatorname{Spec}(R[t^{-1}])$ turn out to be an affine covering of the *blowup* of $\operatorname{Spec}(R)$ along the ideal (z_0, z_1). We'll come back to this point later.

The following corollary of Lemma 7 will be useful on the homework (for proving that finite morphisms are proper). This is basically the *going-up theorem* of commutative algebra (Atiyah-Macdonald, Theorem 5.10).

Corollary 8. *Let R be an integral domain and let $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be a finite morphism. If $S \otimes_R \operatorname{Frac}(R) \neq 0$, then f is surjective.*

The condition that $S \otimes_R \operatorname{Frac}(R) \neq 0$ is equivalent to saying that the map $R \to S$ is injective (as in the proof of Lemma 4).

Proof. By induction on the number of generators, we may assume that $S \cong R[x]/(P)$ for some monic polynomial $P \in R[x]$. But then $\operatorname{Spec}(S)$ admits a closed immersion into $\mathbb{P}^1_R$ (and also $\mathbb{A}^1_R$), so the surjectivity follows from Lemma 7. $\square$

Proof of Theorem 5. It is easy to check directly that the map f is separated (homework) and it is clear that it is of finite type. The difficulty then is to check that f is universally closed; that is, we must check that for any reduced ring R and any closed subscheme Z of $\mathbb{P}^n_R$, the image of $f : Z \to \operatorname{Spec}(R)$ is closed. (Really we are supposed to check images of closed subsets of $\mathbb{P}^n_R$, but by Lemma 1 these all arise from closed subschemes.) Since $Z \to \mathbb{P}^n_R$ is quasicompact (it's a closed immersion) and $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is quasicompact (obvious), by Lemma 4 we need only check that $f(Z)$ is stable under specialization.

Pick any $x, y \in \operatorname{Spec}(R)$ such that y is a specialization of x and $x \in f(Z)$; we need to show that also $y \in f(Z)$. Using Lemma 1, we may replace $\operatorname{Spec}(R)$ with the closure of x, i.e., the quotient $R/\mathfrak{p}$ where $\mathfrak{p}$ is the ideal of R corresponding to x. In this case, R is an integral domain and $x = \operatorname{Spec}(K)$ for $K = \operatorname{Frac}(R)$. In case $n = 1$, Lemma 7 proves the claim. The general case follows by a variant of the same argument, as follows.

Fix homogeneous coordinates on $\mathbb{P}^n_R$. By hypothesis, the restriction of Z to $\mathbb{P}^n_K$ is a nonempty closed subscheme, so it must contain a point z whose residue field is an extension

L of K. (We can make L finite over K by choosing a closed point and invoking the Nullstellensatz, but this is not really necessary.) Choose an L-valued point $z' \in \mathbb{P}^1_L$ lifting z, and write z' in homogeneous coordinates as $[z_0 : \cdots : z_n]$ with $z_0, \dots, z_n \in L$.

Pick $x \in \operatorname{Spec}(R)$. By applying Lemma 7 repeatedly, we construct an R-subalgebra S of L such that x belongs to the image of $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ and for all $i, j \in \{0, \dots, n\}$, either z_i/z_j or z_j/z_i (or both) is contained in S.

Under the partial order by divisibility in S, any two of $z_0, \dots, z_n$ are comparable. There must then be a least element among this set, which we may take to be z_0; then $z_1/z_0, \dots, z_n/z_0 \in S$. As in Lemma 7, we get a section $\operatorname{Spec}(S) \to \mathbb{P}^n_S$ of the projection map whose composition with the map $\mathbb{P}^n_S \to \mathbb{P}^n_R$ has image in Z (because closure of image contains image of closure). Since $\operatorname{Spec}(S)$ covers x, so then must Z. $\square$