Math 203B (Algebraic Geometry), UCSD, winter 2020
Problem Set 1 (due Wednesday, January 15)

Solve the following problems, and turn in the solutions to at least *four* of them.

General note: you are welcome to look things up as you try to solve the exercises, as long as you cite your sources. Typical sources: Gathmann's notes, Hartshorne, Atiyah-Macdonald, Stacks Project, Wikipedia, PlanetMath, MathOverflow, Stack Exchange. (Google is typically *not* a source; cite the originating website.) Likewise, you are welcome to collaborate with other students in the course, as long as you cite these collaborations. (When appropriate, break this down by individual problem.)

Note: whenever we refer to $\mathbb{R}$ or $\mathbb{C}$ as a topological space, we mean the usual topology.

1. Take $X = \mathbb{R}$ and let $\mathcal{F}$ be the presheaf of sets with the property that $\mathcal{F}(U) = \{0, 1\}$ for every open subset U of X. Compute the sheafification of $\mathcal{F}$ (that is, describe its values on an arbitrary open subset of X). Hint: express the answer in terms of functions from U to $\{0, 1\}$.

2. (a) Take $X = \mathbb{C}$ and let $\mathcal{F}$ be the sheaf of holomorphic functions. Is the Taylor series map from the stalk $\mathcal{F}_0$ to the power series ring $\mathbb{C}[\![z]\!]$ injective? Why or why not?

   (b) Take $X = \mathbb{R}$ and let $\mathcal{F}$ be the sheaf of C^∞ functions. Is the Taylor series map from the stalk $\mathcal{F}_0$ to the power series ring $\mathbb{R}[\![z]\!]$ injective? Why or why not?

3. Let X be the unit circle in $\mathbb{R}^2$. Prove that the sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{R}/\mathbb{Z}} \to 0$$

   of sheaves of abelian groups is exact, but taking global sections does not yield an exact sequence. (Note: by $\underline{Y}$ I mean the sheaf of continuous functions to Y, not the locally constant sheaf valued in Y.)

4. Let R be the ring $\mathbb{F}_p[x][x^{1/p^j} : j = 1, 2, \dots]$. Prove that R is not a noetherian ring, but $\operatorname{Spec} R = \operatorname{Spec} \mathbb{F}_p[x]$ is a noetherian topological space.

5. Describe all of the closed points of the topological space $\operatorname{Spec} \mathbb{R}[x, y]$, using the map $\operatorname{Spec} \mathbb{C}[x, y] \to \operatorname{Spec} \mathbb{R}[x, y]$ and the Nullstellensatz.

6. Find an example of a noetherian ring R for which the closed points of $\operatorname{Spec} R$ are not dense.