**Math 203B (Algebraic Geometry), UCSD, winter 2020**
**Problem Set 2 (due Wednesday, January 22)**

Solve the following problems, and turn in the solutions to at least *four* of them.

1. Let $A$ be a ring and let $f$ be an element. Prove that the ring homomorphism $A \to A_f$ induces a map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ which induces an isomorphism $\operatorname{Spec} A_f \to D(f)$ of locally ringed spaces. (Here $D(f)$ is being viewed as an open subspace of $\operatorname{Spec} A$.)

2. Let $A$ be a ring and put $X = \operatorname{Spec} A$. Prove that the ring $A$ is reduced (i.e., its only nilpotent element is zero) if and only if the stalks $\mathcal{O}_{X,x}$ are reduced for all $x \in X$.

3. Put $X = \operatorname{Spec} \mathbb{C}[x, y]$. Consider the three distinguished open sets

$$U_1 = D(x), \qquad U_2 = D(y), \qquad U_3 = D(x + y - 1).$$

   Prove that these form an open covering of $X$, but the union of any two of them is not affine. (Hint: for any open subset $U$ of $X$, there is a natural map $U \to \operatorname{Spec}(R)$ for $R = \mathcal{O}(U)$. If $U$ is affine, this map must be a bijection.)

4. Let $A$ be a ring and put $X = \operatorname{Spec} A$. Prove that there is a bijection between idempotent elements of $A$ (i.e., elements $f \in A$ satisfying $f^2 = f$) and closed-open subsets of $X$.

5. (a) View $\mathbb{C}$ as a locally ringed space equipped with the sheaf of holomorphic functions. Let $\varphi : \mathbb{C} \to \operatorname{Spec} \mathbb{C}[t]$ be the function taking $z \in \mathbb{C}$ to the maximal ideal $(t - z)$. Prove that $\varphi$ is a morphism of locally ringed spaces.

   (b) Repeat (a), but now viewing $\mathbb{C}$ as an algebraic variety.

6. (a) Prove that every scheme admits exactly one morphism to $\operatorname{Spec} \mathbb{Z}$.

   (b) Let $p$ be a prime. Prove that a scheme $X$ admits at most one morphism to $\operatorname{Spec} \mathbb{F}_p$, and that this morphism exists if and only if $p = 0$ in $\mathcal{O}(U)$ for every open subset $U$ of $X$.