Math 203B (Algebraic Geometry), UCSD, winter 2020
Problem Set 3 (due Wednesday, January 29)

Solve the following problems, and turn in the solutions to at least *four* of them.

1. (a) Prove that for any locally ringed space X, there is a unique morphism $X \to \operatorname{Spec} \mathcal{O}(X)$ for which taking global sections yields the identity morphism on $\mathcal{O}(X)$. (Hint: this is similar to the proof from lecture that every morphism of affine schemes comes from a morphism of rings.)

   (b) Let X be the complex-analytic projective n-space with homogeneous coordinates $[x_0 : \cdots : x_n]$. Use (a) to produce a morphism $X \to \mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_n]$ of locally ringed spaces. (Hint: apply (a) not to X itself, but to suitable open subspaces.)

   (c) Repeat (b) with X being the projective n-space in the category of varieties over $\mathbb{C}$.

2. Let R be a ring and let n be a positive integer.

   (a) Prove that $\mathcal{O}(\mathbb{P}^n_R) = R$.

   (b) Using (a), prove that $\mathbb{P}^n_R$ is not affine.

3. Let X be a scheme and put $A = \mathcal{O}_X(X)$. Let $f_1, \dots, f_n \in A$ be elements which generate the unit ideal. For $i = 1, \dots, n$, let X_i be the open subscheme of X consisting of those points x for which f_i does not belong to the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Suppose that X_i is affine for $i = 1, \dots, n$, and put $A_i = \mathcal{O}_X(X_i)$.

   (a) For $i, j = 1, \dots, n$, prove that the open subscheme $X_i \cap X_j$ of X is isomorphic to $\operatorname{Spec}(A_i[f_j^{-1}])$.

   (b) Prove that the natural map $A_{f_i} \to A_i$ is an isomorphism. Hint: start with the exact sequence
   $$0 \to A \to \bigoplus_{i=1}^n A_i \to \bigoplus_{i,j=1}^n A_{ij}$$
   for $A_{ij} = \mathcal{O}_X(X_i \cap X_j)$, then invert f_i.

   (c) Prove that X is isomorphic to $\operatorname{Spec}(A)$.

4. Let M be a module over a ring R. Suppose that $f_1, f_2 \in R$ generate the unit ideal. Prove that the sequence
$$0 \to M \to M_{f_1} \oplus M_{f_2} \to M_{f_1 f_2} \to 0$$
is exact, where the map $M_{f_1} \oplus M_{f_2} \to M_{f_1 f_2}$ is $(x, y) \mapsto x - y$. (Hint: we already know this if $M = R$, and by extension of M is a free module. Now use the fact that every module is a quotient of a free module.)

5. Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a graded ring.

   (a) Prove that for any $m > 0$, the graded ring $S_0 \oplus \bigoplus_{n=m}^{\infty} S_n$ has the same Proj as does S.

   (b) Prove that for any $m > 0$, the graded ring $\bigoplus_{n=0}^{\infty} S_{mn}$ has the same Proj as does S. (This has a geometric meaning which we will come back to later.)

6. Read the Wikipedia entry for "Veronese surface", then describe a morphism of graded rings corresponding to the inclusion $\mathbb{P}^2_R \to \mathbb{P}^5_R$ for R an arbitrary base ring.