Math 203B (Algebraic Geometry), UCSD, winter 2020
Problem Set 4 (due Wednesday, February 12)

Solve the following problems, and turn in the solutions to at least *four* of them.

1. (a) A morphism $f : Y \to X$ of schemes is *surjective* if its underlying map $|Y| \to |X|$ of sets is surjective. Prove that this property is stable under base change.

   (b) Take $X = Y = \operatorname{Spec} \mathbb{C}, S = \operatorname{Spec} \mathbb{R}$. Show that in this case, even though X, Y, S each consist of a single point, $X \times_S Y$ does not.

2. Hartshorne exercise II.3.10.

3. Let $f : Y \to X$ be a morphism of schemes.

   (a) For $\operatorname{Spec} R$ an open affine subscheme of X, prove that the property that $Y \times_X \operatorname{Spec} R$ is of finite type over $\operatorname{Spec} R$ satisfies affine communication.

   (b) Repeat with "of finite type" replaced by "locally of finite type".

4. Hartshorne exercise II.5.2. This gives some examples of $\mathcal{O}_X$-modules on a scheme X which are not quasicoherent.

5. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X. Prove that there exist an affine morphism $f : Y \to X$ of schemes and an isomorphism $f_* \mathcal{O}_Y \cong \mathcal{F}$ of sheaves of $\mathcal{O}_X$-algebras. This morphism is known as the *relative Spec* of $\mathcal{F}$ over X.

6. Let $f : X \to S$ be a morphism of schemes with S affine.

   (a) Prove that if f is separated, the intersection of any two open affine subschemes of X is again affine.

   (b) Prove that the condition of f being separated cannot be omitted from (a), using the example of two affine planes glued along the complement of a point.