Math 203B (Algebraic Geometry), UCSD, winter 2020
Problem Set 7 (due Wednesday, March 4)

Solve the following problems, and turn in the solutions to at least *four* of them. Note that some of the Hartshorne exercises are now from Chapter III rather than Chapter II.

Throughout this assignment, for A a ring and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_A$, let $H^i(\mathbb{P}^n_A, \mathcal{F})$ be the cohomology of the complex

$$0 \to \bigoplus_{0 \le i \le n} \Gamma(U_i, \mathcal{O}(d)) \to \bigoplus_{0 \le i < j \le n} \Gamma(U_i \cap U_j, \mathcal{O}(d)) \to \bigoplus_{0 \le i < j < k \le n} \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}(d)) \to \cdots.$$

1. Let $f : Y \to X$ be a qcqs morphism of schemes. Let $\mathcal{F}$ be a quasicoherent sheaf on X.
   (a) Show that there is a natural map $\mathcal{F} \to f_* f^* \mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules. (Reminder: the qcqs condition is there to guarantee that f_* preserves quasicoherence.)
   (b) Now assume that f is surjective and there is an open dense subset U of X over which f is an isomorphism. Prove that the kernel of $\mathcal{F} \to f_* f^* \mathcal{F}$ is supported on $X \setminus U$.

2. Let $f : X \to S$ be a morphism of schemes with $S = \operatorname{Spec} A$ for some noetherian ring A. Let $\mathcal{F}$ be a coherent sheaf on X.
   (a) It was shown in class that if $X = \mathbb{P}^n_S$, then $\Gamma(X, \mathcal{F})$ is a finite A-module. Deduce the same when X admits a closed immersion into $\mathbb{P}^n_S$.
   (b) Using Chow's lemma and the previous exercise, prove the same when f is proper. (Hint: induct on the dimension of X.)

3. Compute the cohomology of the sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n_A$. There are many references for this (including Hartshorne), but try on your own first!

4. Hartshorne exercise III.5.2(a).

5. Let $\mathcal{F}$ be a locally free coherent sheaf on $\mathbb{P}^1_K$ of rank 2 (for K a field).
   (a) Suppose that that there exists a short exact sequence
   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$
   in which $n_1 \ge n_2$. Prove that the exact sequence splits. Hint: reduce to the case $n_2 = 0$, then use the long exact sequence in cohomology.
   (b) Suppose that there exists a short exact sequence
   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$
   in which $n_1 \le n_2 - 1$. Prove that there also exists a short exact sequence
   $$0 \to \mathcal{O}(n_1 + c) \to \mathcal{F} \to \mathcal{O}(n_2 - c) \to 0$$

for some positive integer c. Hint: this time, reduce to the case $n_1 = -1$ and remember that every line bundle on $\mathbb{P}^1_K$ of degree n is isomorphic to $\mathcal{O}(n)$. But be careful: the quotient of two vector bundles is not always a vector bundle!

6. Let $\mathcal{F}$ be a locally free coherent sheaf on $\mathbb{P}^1_K$ of rank d (for K a field).

   (a) Prove that there exists a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F}$ of $\mathcal{F}$ by vector subbundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to $\mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$. Hint: use the fact that $\mathcal{F}(n)$ is generated by global sections for n sufficiently large.

   (b) Using (a) and the previous exercise, prove that there exists an isomorphism

$$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d)$$

   for some $n_1, \ldots, n_d \in \mathbb{Z}$ (but not necessarily the same ones you found in (a)). Hint: note that the sum of the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ is independent of the filtration. Then use the previous exercise to raise the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for small values of i at the expense of larger values.